Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. Show how to find the distance between the points \( P_0(4, -2, 3) \) and \( P_1(2, -2, 4) \).

Solution: The distance between these points is

\[
|P_0P_1| = \sqrt{(2 - 4)^2 + (-2 - (-2))^2 + (4 - 3)^2} \\
= \sqrt{(-2)^2 + 0^2 + 1^2} \\
= \sqrt{5}.
\]

(a) Find the unit vector obtained by rotating the vector \( \langle 0, 1 \rangle \) through an angle of \( 120^\circ \) counterclockwise about the origin.

(b) Express the vector \( \mathbf{v} = 5\mathbf{i} + \frac{1}{5}\mathbf{j} \) as the product of its magnitude and direction.

\[
5\mathbf{i} + \frac{1}{5}\mathbf{j} = \underbrace{\left( \begin{array}{c} \text{magnitude} \\ \text{direction} \end{array} \right)}_{\text{vector}}.
\]

Solutions: For part a, notice that the vector \( \langle 0, 1 \rangle \) is already a unit vector that points to the north (in the \( xy \) plane). If we rotate this vector \( 120^\circ \) counterclockwise, we get a unit vector that points \( 30^\circ \) south of west. Based on our knowledge of the unit circle, this vector is \( \langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle \).

For part b, we see that the magnitude of the given vector is

\[
|\mathbf{v}| = \sqrt{5^2 + \left( \frac{1}{5} \right)^2} = \frac{\sqrt{626}}{5}
\]

Thus the direction of this vector is

\[
\frac{1}{|\mathbf{v}|} \mathbf{v} = \frac{5}{\sqrt{626}} \left( 5\mathbf{i} + \frac{1}{5}\mathbf{j} \right) = \frac{25}{\sqrt{626}} \mathbf{i} + \frac{1}{\sqrt{626}} \mathbf{j}.
\]

Therefore

\[
\mathbf{v} = \frac{\sqrt{626}}{5} \underbrace{\left( \begin{array}{c} \text{magnitude} \\ \text{direction} \end{array} \right)}_{\text{vector}}.
\]
2. Let \( \mathbf{u} \) and \( \mathbf{v} \) be the vectors
\[
\mathbf{u} = -\sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} \\
\mathbf{v} = \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + 2\mathbf{k}
\]
and let \( \theta \) be the angle between \( \mathbf{u} \) and \( \mathbf{v} \).

(a) Find \( \mathbf{u} \cdot \mathbf{v} \), \( |\mathbf{u}| \), and \( |\mathbf{v}| \).
(b) Find \( \cos (\theta) \).
(c) Find the projection of \( \mathbf{u} \) onto \( \mathbf{v} \) (denoted by \( \text{proj}_\mathbf{v} \mathbf{u} \)).
(d) Find the scalar component of \( \mathbf{u} \) in the direction of \( \mathbf{v} \).

Solution:
\[
\mathbf{u} \cdot \mathbf{v} = \left( -\sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} \right) \cdot \left( \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + 2\mathbf{k} \right) \\
= \left( -\sqrt{2} \right) \left( \sqrt{2} \right) + \left( \sqrt{3} \right) \left( \sqrt{3} \right) + \left( 0 \right) \left( 2 \right) \\
= 1
\]
and
\[
|\mathbf{u}| = \sqrt{(\sqrt{2})^2 + (\sqrt{3})^2} = \sqrt{5} \\
|\mathbf{v}| = \sqrt{(\sqrt{2})^2 + (\sqrt{3})^2 + 2^2} = 3.
\]

Thus
\[
\cos (\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} = \frac{1}{3\sqrt{5}} = \frac{\sqrt{5}}{15},
\]
\[
\text{proj}_\mathbf{v} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = \frac{1}{9} \left( \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + 2\mathbf{k} \right),
\]
and the scalar component of \( \mathbf{u} \) in the direction of \( \mathbf{v} \) is
\[
\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{1}{3}.
\]

3. Which of the following statements are always true and which are not always true? (Circle the correct choice.)

(a) \( |\mathbf{v}| = \mathbf{v} \cdot \mathbf{v} \) (always true, not always true)
(b) \( \mathbf{u} \cdot \mathbf{v} = - (\mathbf{v} \cdot \mathbf{u}) \) (always true, not always true)
(c) \( (3\mathbf{u}) \cdot \mathbf{v} = 3 (\mathbf{u} \cdot \mathbf{v}) \) (always true, not always true)
(d) \( \mathbf{u} \times \mathbf{0} = \mathbf{0} \) (always true, not always true)
(e) \( \mathbf{u} \times (-\mathbf{u}) = \mathbf{0} \) (always true, not always true)
(f) \( \mathbf{u} \times \mathbf{v} = - (\mathbf{v} \times \mathbf{u}) \) \( \text{(always true, not always true)} \)

(g) \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \) (always true, \( \text{not always true} \))

(h) \( (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = |\mathbf{u} \times \mathbf{v}| |\mathbf{v}| \) (always true, \( \text{not always true} \))

(i) \( |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \cos (\theta) \) where \( \theta \) is the angle between \( \mathbf{u} \) and \( \mathbf{v} \). (always true, \( \text{not always true} \))

(j) \( \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos (\theta) \) where \( \theta \) is the angle between \( \mathbf{u} \) and \( \mathbf{v} \). (\text{always true}, \( \text{not always true} \))

Grading of this question will be as follows:

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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>2</td>
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<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
</tr>
</tbody>
</table>

4. Find either parametric equations of symmetric equations of the line that contains the point \( P_0 (-4, 2, -4) \) and is perpendicular to the plane \( 3x + 3y + z = -1 \).

**Solution:** The vector \( (3, 3, 1) \) is perpendicular to the given plane and is thus parallel to the line in question. Hence this is a direction vector for the line in question. A vector equation for the line is

\[
(x, y, z) = (-4, 2, -4) + t (3, 3, 1).
\]

Parametric equations are

\[
x = -4 + 3t \\
y = 2 + 3t \\
z = -4 + t \\
-\infty < t < \infty.
\]

5. The position vector of a particle moving in space is given by

\[
\mathbf{r} (t) = (t + 1) \mathbf{i} + (t^2 - 1) \mathbf{j} + 2t \mathbf{k}.
\]

(a) Find the velocity vector, \( \mathbf{v} (t) \).

(b) Find the acceleration vector, \( \mathbf{a} (t) \).

(c) Find the speed, \( \mathbf{v} (t) \).

(d) Find parametric equations for the line that is tangent to the curve of motion at the point on the curve corresponding to time \( t = 1 \).

**Solution:**

\[
\mathbf{v} (t) = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t \mathbf{j} + 2 \mathbf{k}
\]

and

\[
\mathbf{a} (t) = \frac{d\mathbf{v}}{dt} = 2 \mathbf{j}
\]

and

\[
\mathbf{v} (t) = |\mathbf{v} (t)| = \sqrt{1^2 + (2t)^2 + 2^2} = \sqrt{4t^2 + 5}.
\]
At time $t = 1$ we have $\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and this vector is tangent to the curve of motion at time $t = 1$. Also the point on the curve of motion that corresponds to time $t = 1$ is $(2, 0, 2)$. Thus the tangent line at this point is given by the parametric equations

$$
\begin{align*}
x &= 2 + t \\
y &= 0 + 2t \\
z &= 2 + 2t \\
-\infty < t < \infty.
\end{align*}
$$

6. Please be as detailed as possible in answering this question (so that I can follow your reasoning easily):

A projectile is fired with an initial speed of $500 \text{ m/s}$ at an angle of elevation of $45^\circ$.

(a) Begin with the acceleration function $\mathbf{a}(t) = -g\mathbf{j}$ (where $g = 9.8 \text{ m/s}^2$) and show the steps involved in obtaining the velocity function, $\mathbf{v}(t)$, and the position function, $\mathbf{r}(t)$.

(b) Show how to determine when and how far away (from where it is fired) the projectile will strike the ground.

**Solution:** Since

$$
\mathbf{a}(t) = -g\mathbf{j},
$$

then

$$
\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}(0).
$$

Also

$$
\mathbf{v}(0) = 500 \cos(45^\circ) \mathbf{i} + 500 \sin(45^\circ) \mathbf{j} = 250\sqrt{2}\mathbf{i} + 250\sqrt{2}\mathbf{j}
$$

and thus

$$
\mathbf{v}(t) = 250\sqrt{2}\mathbf{i} + \left(250\sqrt{2} - gt\right)\mathbf{j}.
$$

Therefore

$$
\mathbf{r}(t) = 250\sqrt{2}t\mathbf{i} + \left(250\sqrt{2}t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{r}(0)
$$

and since $\mathbf{r}(0) = \mathbf{0}$ we obtain

$$
\mathbf{r}(t) = 250\sqrt{2}t\mathbf{i} + \left(250\sqrt{2}t - \frac{1}{2}gt^2\right)\mathbf{j}.
$$

In parametric form, the $x$ and $y$ components of the path of motion are

$$
\begin{align*}
x(t) &= 250\sqrt{2}t \\
y(t) &= 250\sqrt{2}t - \frac{1}{2}gt^2
\end{align*}
$$

where $T_f$ is the time that the projectile lands.
To find $T_f$, we solve $y(t) = 0$ for $t$: Setting

$$250\sqrt{2}t - \frac{1}{2}gt^2 = 0,$$

we obtain either $t = 0$ (obviously not $T_f$) and

$$\frac{1}{2}gt = 250\sqrt{2}$$

or

$$t = T_f = \frac{500\sqrt{2}}{g} \approx 72 \text{ seconds}.$$

The range of the projectile is

$$x(T_f) = 250\sqrt{2}T_f$$

$$= 250\sqrt{2} \left( \frac{500\sqrt{2}}{g} \right)$$

$$= \frac{500^2}{g}$$

$$\approx 25,510 \text{ meters}$$

(or about 25.51 kilometers).