1. Draw contour maps of the functions given in parts a–d, showing several (at least five or six) level curves for each.

(a) \( f(x, y) = \sqrt{x + y} \)
These level curves all have the form \( \sqrt{x + y} = k \) or \( x + y = k^2 \) of \( y = -x + k^2 \)
The level curves are parallel lines with slope \(-1\). None of them lie below the line \( y = -x \).

(b) \( f(x, y) = x^2 + y^2 \)
These level curves all have the form \( x^2 + y^2 = k \) (where \( k \geq 0 \)). The level curves are circles centered at the origin. (Actually, the “curve” that corresponds to \( k = 0 \) is just the origin itself.)
(c) \( f(x, y) = y - x^2 \)
These level curves all have the form \( y - x^2 = k \) or \( y = x^2 + k \). They are parabolas.

\[ \begin{align*}
\end{align*} \]

(d) \( f(x, y) = y^2 - x^2 \)
These level curves all have the form \( y^2 - x^2 = k \) or \( y^2 = x^2 + k \) or \( y = \pm \sqrt{x^2 + k} \). The level curves are hyperbolas.

\[ \begin{align*}
\end{align*} \]

2. Evaluate each of the limits in parts a and b. If the limit does not exist, then explain why. If the limit does exist, then find it and use the Squeeze Theorem to prove that the limit exists.

(a) \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} \)
This limit does not exist because

\[ \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{x}{x^2 + 0^2} = \lim_{x \to 0} \frac{0}{x^2} = 0 \]
along the \( x \) axis

and

\[ \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2}{x^2 + x^2} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2} \]
along the line \( y = x \)
(b) 

\[
\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}
\]

This limit does exist and is equal to 0. Here is why: For any point \((x, y) \neq (0, 0)\), we have

\[
\left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = \frac{|x|}{\sqrt{x^2 + y^2}} |y|.
\]

Also,

\[x^2 \leq x^2 + y^2\]

so

\[|x| \leq \sqrt{x^2 + y^2}\]

and thus

\[
\frac{|x|}{\sqrt{x^2 + y^2}} \leq 1.
\]

This implies that

\[
\left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = \frac{|x|}{\sqrt{x^2 + y^2}} |y| \leq 1 \cdot |y|
\]

and hence that

\[-|y| \leq \frac{xy}{\sqrt{x^2 + y^2}} \leq |y|
\]

for all points \((x, y) \neq (0, 0)\).

Since

\[
\lim_{(x,y)\to(0,0)} |y| = 0 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} (-|y|) = 0,
\]

the Squeeze Theorem tells us that

\[
\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0.
\]

3. Find the first partial derivatives \((f_x \text{ and } f_y)\) of the functions given in parts a–d.

(a) \(f(x,y) = 3x - 2y^4\)

\[
f_x = 3 \quad f_y = -8y^3
\]

(b) \(f(x,y) = xe^{3y}\)

\[
f_x = (x) \frac{\partial}{\partial x} (e^{3y}) + (e^{3y}) \frac{\partial}{\partial x} (x)
\]  
\[= (x)(0) + (e^{3y})(1)
\]  
\[= e^{3y}
\]
and

\[ f_y = (x) \frac{\partial}{\partial y} (e^{3y}) + (e^{3y}) \frac{\partial}{\partial y} (x) \]
\[ = (x) (3e^{3y}) + (e^{3y}) (0) \]
\[ = 3xe^{3y} \]

(c) \( f(x, y) = \cos (x - 3y^2) \)

\[ f_x = -\sin (x - 3y^2) (1) = -\sin (x - 3y^2) \]
\[ f_y = -\sin (x - 3y^2) (-6y) = 6y \sin (x - 3y^2) \]

(d) \( f(x, y) = e^{x-y} \cos (x) \)

\[ f_x = (e^{x-y}) \frac{\partial}{\partial x} (\cos (x)) + (\cos (x)) \frac{\partial}{\partial x} (e^{x-y}) \]
\[ = (e^{x-y}) (-\sin (x)) + (\cos (x)) (e^{x-y}) \]
\[ = e^{x-y} (\cos (x) - \sin (x)) \]

and

\[ f_y = (e^{x-y}) \frac{\partial}{\partial y} (\cos (x)) + (\cos (x)) \frac{\partial}{\partial y} (e^{x-y}) \]
\[ = (e^{x-y}) (0) + (\cos (x)) (-e^{x-y}) \]
\[ = -e^{x-y} \cos (x) \]

4. Find an equation of the tangent plane to the surface

\[ z = e^{x^2-y^2} \]

at the point \((1, -1, 1)\).

\textbf{Solution:}

\[ \frac{\partial z}{\partial x} = 2xe^{x^2-y^2} \]

and

\[ \frac{\partial z}{\partial y} = -2ye^{x^2-y^2} \]

so

\[ \frac{\partial z}{\partial x} (1, -1) = 2 \]

and

\[ \frac{\partial z}{\partial y} (1, -1) = 2. \]

Therefore, the tangent plane to the surface \( z = e^{x^2-y^2} \) at the point \((1, -1, 1)\) has equation

\[ z - 1 = 2(x - 1) + 2(y + 1) \]

or

\[ 2x + 2y - z = -1. \]
5. Given that
\[ u = xy + yz + zx \]
\[ x = st \]
\[ y = e^{st} \]
\[ z = t^2, \]
find \( \partial u / \partial s \) and \( \partial u / \partial t \).

**Solution:**

\[
\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}
= (y + z) t + (x + z) te^{st} + (y + x) (0)
\]
and

\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t}
= (y + z) s + (x + z) se^{st} + (y + x) (2t) .
\]

6. Find the directional derivative of the function
\[ f(x, y) = x^2y - y - 2 \]
at the point \((-3, 2)\) in the direction of the unit vector
\[ u = \frac{\sqrt{2}}{2} i - \frac{\sqrt{2}}{2} j. \]

**Solution:** Since
\[ \nabla f(x, y) = 2xy i + (x^2 - 1) j, \]
we have
\[ \nabla f(-3, 2) = -12i + 8j. \]
Therefore
\[
D_u f(-3, 2) = \nabla f(-3, 2) \cdot u
= -12 \left( \frac{\sqrt{2}}{2} \right) + 8 \left( -\frac{\sqrt{2}}{2} \right)
= -6\sqrt{2} - 4\sqrt{2}
= -10\sqrt{2}.
\]

7. Four statements (A, B, C, and D) are given below. Answer the questions (by circling the correct answer) in parts a–d.

**Statement A:** The function \( f \) is continuous at the point \((a, b)\).
Statement B: The partial derivatives, \( f_x(a, b) \) and \( f_y(a, b) \), both exist.

Statement C: The partial derivative functions, \( f_x \) and \( f_y \), are both continuous at the point \( (a, b) \).

Statement D: The function \( f \) is differentiable at the point \( (a, b) \).

Circle all (if any) that apply.

(a) If Statement A is true, then
   1. statement B must be true. (No)
   2. statement C must be true. (No)
   3. statement D must be true. (No)

(b) If Statement B is true, then
   1. statement A must be true. (No)
   2. statement C must be true. (No)
   3. statement D must be true. (No)

(c) If Statement C is true, then
   1. statement A must be true. (Yes)
   2. statement B must be true. (Yes)
   3. statement D must be true. (Yes)

(d) If Statement D is true, then
   1. statement A must be true. (Yes)
   2. statement B must be true. (Yes)
   3. statement C must be true. (No)