1. Match each of the functions (a and b) with its graph (labelled A and B) and with its contour map (labelled I and II). Write a brief explanation (a few sentences) explaining some of the reasons for your answers.

(a) $z = (1 - x^2)(1 - y^2)$ matches A and I.
(b) $z = (1 - x)^2(1 - y)^2$ matches B and II.
2. Show that
\[ \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} = 2. \]

*Hint:* Use polar coordinates.

**Solution:** If we use polar coordinates, \( x = r \cos(\theta), \ y = r \sin(\theta) \), then our problem becomes
\[
\lim_{r \to 0} \frac{r^2}{\sqrt{r^2 + 1} - 1}
\]
because letting \((x, y)\) approach \((0, 0)\) is the same as letting \( r \) approach 0. Since
\[
\frac{r^2}{\sqrt{r^2 + 1} - 1} = \frac{r^2}{\sqrt{r^2 + 1} - 1} \cdot \frac{\sqrt{r^2 + 1} + 1}{\sqrt{r^2 + 1} + 1}
\]
\[
= \frac{r^2}{(r^2 + 1) - 1}
\]
\[
= \frac{r^2}{r^2 + 1 + 1}
\]
we see that
\[
\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} = \lim_{r \to 0} \frac{r^2}{\sqrt{r^2 + 1} - 1} = \lim_{r \to 0} \left( \frac{\sqrt{r^2 + 1} + 1}{r^2} \right) = 2.
\]

3. Find the first and second partial derivatives of the function
\[ f(x, y) = e^{x/y}. \]
4. Find the linearization, \( L(x, y) \), of the function
\[
f(x, y) = \sqrt{x + e^{4y}}
\]
at the point \((3, 0)\).

**Solution:**
\[
f_x = \frac{1}{2} (x + e^{4y})^{-1/2} = \frac{1}{2\sqrt{x + e^{4y}}}
\]
and
\[
f_y = \frac{1}{2} (x + e^{4y})^{-1/2} (e^{4y}) (4) = \frac{2e^{4y}}{\sqrt{x + e^{4y}}}
\]
so
\[
f_x (3, 0) = \frac{1}{2\sqrt{3 + e^{4(0)}}} = \frac{1}{4}
\]
and
\[
f_y (3, 0) = \frac{2e^{4(0)}}{\sqrt{3 + e^{4(0)}}} = 1.
\]
Thus the linearization of \( f \) at \((3, 0)\) is
\[
L(x, y) = f(3, 0) + f_x(3, 0)(x - 3) + f_y(3, 0)(y - 0)
\]
\[
= 2 + \frac{1}{4} (x - 3) + y
\]
\[
= \frac{1}{4} x + y + 1.25.
\]
5. For the function
\[ z = \ln \left( x^2 + y^2 \right) \]
where
\[ x = s + 2t \]
and
\[ y = 2s - t, \]
use the Chain Rule to find \( \partial z/\partial s \) and \( \partial z/\partial t \).

**Solution:**
\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = \left( \frac{2x}{x^2 + y^2} \right) (1) + \left( \frac{2y}{x^2 + y^2} \right) (2)
\]
\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \left( \frac{2x}{x^2 + y^2} \right) (2) + \left( \frac{2y}{x^2 + y^2} \right) (-1).
\]

6. The length, \( L \), width \( W \), and height \( H \), of a rectangular box change with time. At a certain instant, the dimensions are \( L = 1 \text{ ft} \), \( W = 2 \text{ ft} \), and \( H = 2 \text{ ft} \). At this same instant \( L \) and \( W \) are each increasing at the rate of \( 2 \text{ ft/min} \) and \( H \) is decreasing at the rate of \( 3 \text{ ft/min} \). Find the rate at which the volume of the box is changing at this instant. (You must be detailed in your solution.)

**Solution:** The volume of the box is
\[ V = LWH. \]

Since \( L \), \( W \), and \( H \) are each functions of time \( (t) \), we have (by two applications of the Product Rule - and by using the Chain Rule) that
\[
\frac{\partial V}{\partial t} = \frac{\partial}{\partial t} (LWH) = L \frac{\partial}{\partial t} (WH) + (WH) \frac{\partial L}{\partial t} = L \left( W \frac{\partial H}{\partial t} + H \frac{\partial W}{\partial t} \right) + WH \frac{\partial L}{\partial t} = LW \frac{\partial H}{\partial t} + LH \frac{\partial W}{\partial t} + WH \frac{\partial L}{\partial t}.
\]

At the particular instant (which we will call \( T \)) in question, we see that the volume of the box is changing at the rate
\[
\left. \frac{\partial V}{\partial t} \right|_{t=T} = (1) (2) (-3) + (1) (2) (2) + (2) (2) (2) = 6 \text{ ft}^3/\text{min}.
\]

7. Find the directional derivative of the function
\[ f(x, y) = \sqrt{5x - 4y} \]
at the point \((4, 1)\) in the direction of the unit vector \( u = \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \). (You must of course include all details of your solution.)
Solution:

\[ \frac{\partial f}{\partial x} = \frac{5}{2\sqrt{5x - 4y}} \]

and

\[ \frac{\partial f}{\partial y} = \frac{-2}{\sqrt{5x - 4y}} \]

so

\[ \frac{\partial f}{\partial x} \bigg|_{(x,y)=(4,1)} = \frac{5}{2\sqrt{5(4) - 4(1)}} = \frac{5}{8} \]

and

\[ \frac{\partial f}{\partial y} \bigg|_{(x,y)=(4,1)} = \frac{-2}{\sqrt{5(4) - 4(1)}} = -\frac{1}{2} \]

Thus

\[ \nabla f(4,1) = \left\langle \frac{5}{8}, -\frac{1}{2} \right\rangle \]

and

\[ D_{u}f(4,1) = \nabla f(4,1) \cdot u = \left\langle \frac{5}{8}, -\frac{1}{2} \right\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle = \left( \frac{5}{8} \right) \left( \frac{\sqrt{3}}{2} \right) + \left( -\frac{1}{2} \right) \left( -\frac{1}{2} \right) = \frac{5\sqrt{3}}{16} + \frac{1}{4}. \]

8. Find the point on the plane

\[ x - y + z = 4 \]

that is closest to the point \((1,2,3)\). (Be detailed in your solution.)

Solution: The distance from any point, \((x,y,z)\), on the given plane to the point \((1,2,3)\) is

\[ \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} \]

and since

\[ z - 3 = -x + y + 1, \]

this distance is equal to

\[ \sqrt{(x-1)^2 + (y-2)^2 + (-x + y + 1)^2}. \]

It is easier to find the minimum value of the square of the distance (since then we don’t have to deal with square roots). Thus we want to minimize the function

\[ f(x,y) = (x-1)^2 + (y-2)^2 + (-x + y + 1)^2. \]

Since

\[ f_x = 2(x-1) - 2(-x + y + 1) = 2(2x - y - 2) = 4x - 2y - 4 \]

and

\[ f_y = 2(y-2) + 2(-x + y + 1) = -2x + 4y + 2, \]
then any critical point of \( f \) must satisfy the system of equations
\[
\begin{align*}
4x - 2y &= 4 \\
-2x + 4y &= -2.
\end{align*}
\]

The only solution of this system, and hence the only critical point of \( f \), is \((x, y) = (1, 0)\).

It is clear from the nature of this problem that the problem must have a solution and hence that the answer to our question must be that the point on the given plane that is closest to the point \((1, 2, 3)\) is the point \((1, 0, 3)\). However, we can use the Second Derivative Test to make sure that our solution does give a local (and in fact absolute) minimum value of \( f \). Since
\[
\begin{align*}
f_{xx} &= 4 \\
f_{yy} &= 4 \\
f_{xy} &= -2,
\end{align*}
\]
we see that
\[
D = f_{xx}f_{yy} - (f_{xy})^2 = 12 > 0
\]
and
\[
f_{xx} > 0.
\]
The Second Derivative Test then tells us that we do indeed have a local minimum.

9. Suppose that \( f \) is a differentiable function of two variables and suppose that
\[
r (t) = x (t) \mathbf{i} + y (t) \mathbf{j}
\]
is a level curve of \( f \). Prove that, at any given point on this level curve, the gradient vector of \( f \) is orthogonal to the level curve. (When we say that the gradient vector is orthogonal to the level curve, we really mean that the gradient vector is orthogonal to the tangent vector to the level curve - so this is what you should prove.)

\textit{Hint to get started:} Since \( r \) is a level curve of \( f \), we know that the \( f (x(t), y(t)) \) is equal to some constant value for all values of \( t \).

\textbf{Proof:} Since the value of \( f \) is constant along the level curve, we know that
\[
\frac{d}{dt} (f (x(t), y(t))) = 0
\]
for all \( t \). However, by the Chain Rule, we also know that
\[
\frac{d}{dt} (f (x(t), y(t))) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.
\]
Therefore
\[
\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 0
\]
for all \( t \).
In addition, we know that
\[ \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \]
and we also know that the tangent vector to each point on the level curve is given by
\[ \mathbf{r}'(t) = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j}. \]
This means that
\[ \nabla f \cdot \mathbf{r}'(t) = 0 \]
for all values of \( t \) and hence means that \( \nabla f \) is (at each point on the level curve) orthogonal to \( \mathbf{r}'(t) \).

10. Let \( D \) be the set
\[ D = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}. \]
Find the absolute maximum and absolute minimum values of the function
\[ f(x, y) = (1 - x^2)(1 - y^2) \]
on the set \( D \). Also state at which points in \( D \) these absolute extreme values occur. (Be detailed).

**Solution:** Since \( D \) is a rectangle and since either \( x = \pm 1 \) or \( y = \pm 1 \) at any point on the boundary of this rectangle, it is easily seen that \( f(x, y) = 0 \) at all points on the boundary of \( D \). As for the critical points of \( f \), we have
\[ f_x = -2x(1 - y^2) \]
\[ f_y = -2y(1 - x^2) \]
from which we can conclude (by simultaneously solving the equations \( f_x = 0 \), \( f_y = 0 \)) that the only critical point of \( f \) is \((0, 0)\).

Since \( f(0, 0) = 1 \), we conclude that the absolute maximum value of \( f \) on \( D \) is 1 and that this absolute maximum value occurs only at the point \((0, 0)\). Furthermore, the absolute minimum value of \( f \) on \( D \) is 0 and this absolute minimum value of \( f \) occurs at every boundary point of \( D \).

Figure A of problem 1 of this exam shows the graph of \( f \). The plane drawn in Figure A illustrates our conclusion in the present problem.