Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. Suppose that \( f \) is a constant function of two variables, \( f(x, y) = k \). Suppose also that \( R \) is a rectangle, \( R = \{(x, y) \mid a \leq x \leq b, \ c \leq y \leq d\} \). Show that

\[
\iiint_R f(x, y) \, dA = k(b - a)(d - c).
\]

Solution:

\[
\iiint_R f(x, y) \, dA = \int_c^d \int_a^b k \, dA
\]

\[
= k \int_c^d \int_a^b \, dx \, dy
\]

\[
= k \int_c^d \left( x \bigg|^{b}_{x=a} \right) \, dy
\]

\[
= k \int_c^d (b - a) \, dy
\]

\[
= k (b - a) \int_c^d \, dy
\]

\[
= k (b - a) \left( y \bigg|^{d}_{y=c} \right)
\]

\[
= k (b - a) (d - c).
\]

2. Use integration to show that the volume of the solid bounded by the surface \( z = x\sqrt{x^2 + y} \) and by the planes \( x = 0, \ x = 1, \ y = 0, \) and \( y = 1 \) is \( \frac{\sqrt{2}}{15} (4\sqrt{2} - 1) \).
Solution: The volume of this solid is

$$\int \int_D x \sqrt{x^2 + y} \, dA$$

where $D = [0, 1] \times [0, 1]$. Thus the volume is

$$\int_0^1 \int_0^1 x \sqrt{x^2 + y} \, dx \, dy.$$

First we evaluate the inner integral using the substitution $u = x^2 + y$ (for which $du = 2x \, dx$):

$$\int_0^1 x \sqrt{x^2 + y} \, dx = \frac{1}{2} \int_0^{1+y} u^{1/2} \, du = \frac{1}{3} \left( (1 + y)^{3/2} - y^{3/2} \right).$$

This tells us that the volume is

$$\frac{1}{3} \int_0^1 \left( (1 + y)^{3/2} - y^{3/2} \right) \, dy = \frac{1}{3} \left( \int_0^1 (1 + y)^{3/2} \, dy - \int_0^1 y^{3/2} \, dy \right)$$

$$= \frac{1}{3} \left( \frac{2}{5} (1 + y)^{5/2} \bigg|_{y=0}^{y=1} - \frac{2}{5} y^{5/2} \bigg|_{y=0}^{y=1} \right)$$

$$= \frac{2}{15} \left( (2^{5/2} - 1) - (1 - 0) \right)$$

$$= \frac{4}{15} \left( 2\sqrt{2} - 1 \right).$$

3. Show that

$$\int_0^1 \int_x^{2-x} (x^2 - y) \, dy \, dx = \frac{-5}{6}.$$
Solution: First we do the inner integral:

\[
\int_x^{2-x} (x^2 - y) \, dy = x^2 y - \frac{1}{2} y^2 \bigg|_{y=x}^{y=2-x} \\
= \left( x^2 (2 - x) - \frac{1}{2} (2 - x)^2 \right) - \left( x^3 - \frac{1}{2} x^2 \right) \\
= 2x^2 - x^3 - \frac{1}{2} (4 - 4x + x^2) - x^3 + \frac{1}{2} x^2 \\
= 2x^2 - x^3 - 2 + 2x - \frac{1}{2} x^2 - x^3 + \frac{1}{2} x^2 \\
= -2x^3 + 2x^2 + 2x - 2.
\]

We now obtain

\[
\int_0^1 \int_x^{2-x} (x^2 - y) \, dy \, dx = \int_0^1 (-2x^3 + 2x^2 + 2x - 2) \, dx \\
= -2 \left( \frac{1}{4} x^4 - \frac{1}{3} x^3 - \frac{1}{2} x^2 + x \right) \bigg|_{x=0}^{x=1} \\
= -2 \left( \frac{1}{4} - \frac{1}{3} - \frac{1}{2} + 1 \right) \\
= -\frac{5}{6}.
\]

4. Use integration to show that the volume of a sphere of radius \( R \) is \( \frac{4}{3} \pi R^3 \).

Solution: See class notes.

5. Find the mass and the center of mass of the lamina, \( D \), with density function \( \rho(x, y) = y \) that is bounded by the curves \( y = e^x \), \( y = 0 \), \( x = 0 \), and \( x = 1 \). This lamina is pictured below.
Solution: The mass of this lamina is

\[ m = \iint_D \rho(x, y) \, dA = \iint_D y \, dA = \int_0^1 \int_0^{e^x} y \, dy \, dx = \frac{1}{2} \int_0^1 y^2 \bigg|_{y=0}^{y=e^x} \, dx = \frac{1}{2} \int_0^1 e^{2x} \, dx = \frac{1}{4} e^{2x} \bigg|_{x=1}^{x=0} = e^2 - 1. \]

Also

\[ I_y = \iint_D x \rho(x, y) \, dA = \iint_D xy \, dA = \int_0^1 \int_0^{e^x} xy \, dy \, dx = \frac{1}{2} \int_0^1 xy^2 \bigg|_{y=0}^{y=e^x} \, dx = \frac{1}{2} \int_0^1 xe^{2x} \, dx = \frac{1}{4} \left( e^{2x} \left( x - \frac{1}{2} \right) \right) \bigg|_{x=0}^{x=1} = \frac{1}{4} \left( \frac{1}{2} e^2 + \frac{1}{2} \right) = \frac{e^2 + 1}{8}. \]
and

\[ I_x = \int \int_D y \rho(x, y) \, dA = \int \int_D y^2 \, dA \]
\[ = \int_0^1 \int_0^{e^x} y^2 \, dy \, dx \]
\[ = \frac{1}{3} \int_0^1 y^3 \bigg|_{y=0}^{y=e^x} \, dx \]
\[ = \frac{1}{3} \int_0^1 e^{3x} \, dx \]
\[ = \frac{1}{9} e^{3x} \bigg|_{x=0}^{x=1} \]
\[ = e^3 - 1. \]

We conclude that the center of mass of this lamina is at the point \((\bar{x}, \bar{y})\) where

\[ \bar{x} = \frac{I_y}{m} = \frac{e^2 + 1}{8} \cdot \frac{e^2 - 1}{4} = \frac{e^2 + 1}{2(e^2 - 1)} \approx 0.6565 \]

and

\[ \bar{y} = \frac{I_x}{m} = \frac{e^3 - 1}{9} \cdot \frac{e^2 - 1}{4} = \frac{4(e^3 - 1)}{9(e^2 - 1)} \approx 1.3277. \]