Instructions. Your work on this exam will be graded according to two criteria: mathematical correctness and clarity of presentation. In other words, you must know what you are doing (mathematically) and you must also express yourself clearly. In particular, write answers to questions using correct notation and using complete sentences where appropriate. Also, you must supply sufficient detail in your solutions (relevant calculations, written explanations of why you are doing these calculations, etc.). It is not sufficient to just write down an “answer” with no explanation of how you arrived at that answer. As a rule of thumb, the harder that I have to work to interpret what you are trying to say, the less credit you will get. You may use your calculator but you may not use any books or notes.

1. Estimate the volume of the solid that lies below the surface \( z = y^2 \) and above the rectangle

   \[ R = \{ (x, y) \mid 0 \leq x \leq 4, \, -2 \leq y \leq 2 \} \]

   by using a Riemann sum with \( m = 2 \) and \( n = 4 \) and taking midpoints of subrectangles as sample points. Your solution to this should include a drawing of the rectangle \( R \) showing the subdivisions and sample points. **Note:** Do not compute an integral! The point of this problem is to obtain an estimate of the volume by using a Riemann sum.

   **Solution:** The rectangle \( R \) with its subdivisions and sample points is shown below.

   ![Diagram of the rectangle R with sample points]

   Here we have \( \Delta x = \frac{4 - 0}{2} = 2 \), \( \Delta y = \frac{2 - (-2)}{4} = 1 \), and \( \Delta A = \Delta x \Delta y = 2 \). The Riemann sum estimate of the volume (using midpoints) is

   \[
   f (1, -1.5) + f (3, -1.5) + f (1, -0.5) + f (3, -0.5) \\
   + f (1, 0.5) + f (3, 0.5) + f (1, 1.5) + f (3, 1.5)
   \]

   (where \( f \) is the function \( f (x, y) = y^2 \)) all multiplied by \( \Delta A \). Thus the estimate is

   \[
   2 (2.25 + 2.25 + 0.25 + 0.25 + 0.25 + 0.25 + 2.25 + 2.25) = 20.
   \]
2. Use an iterated integral to compute the **exact** volume of the solid described in problem 1.

**Solution:** The exact volume is

\[
\int \int_{R} f(x, y) \, dA = \int_{-2}^{2} \int_{0}^{4} y^2 \, dx \, dy
\]

\[
= \int_{-2}^{2} y^2 \, dx \bigg|_{x=0}^{x=4} \, dy
\]

\[
= \int_{-2}^{2} 4y^2 \, dy
\]

\[
= \frac{4}{3} y^3 \bigg|_{y=-2}^{y=2}
\]

\[
= \frac{4}{3} (16)
\]

\[
= \frac{64}{3}
\]

\[
= \frac{21.3}{3}.
\]

3. Find the volume of the solid under the plane \(x - z = 0\) and above the region, \(D\), bounded by the curves \(y = 0\), \(x = 1\), and \(y = \sqrt{x}\). Be detailed. Your solution must include a picture of the region \(D\) in order to receive full credit. (The answer to this problem is that the volume is \(2/5\).)

**Solution:** The domain \(D\) is pictured below. This is a Type I region.

![Diagram of region D](image)

The volume of the solid being described is

\[
\int \int_{D} x \, dA = \int_{0}^{1} \int_{0}^{\sqrt{x}} x \, dy \, dx.
\]

First we do the inner integral:

\[
\int_{0}^{\sqrt{x}} x \, dy = xy \bigg|_{y=0}^{y=\sqrt{x}} = x^{3/2}.
\]
We now have
\[
\int_0^1 \int_0^{\sqrt{x}} x \, dy \, dx = \int_0^1 x^{3/2} \, dx \\
= \frac{2}{5} x^{5/2} \bigg|_{x=0}^{x=1} \\
= \frac{2}{5}.
\]

4. Let \( D \) be the region pictured below. (The inner boundary of \( D \) is the upper half of the unit circle and the outer boundary is the upper half of the circle of radius 2 centered at the origin.)

Use polar coordinates to evaluate the double integral
\[
\iint_D \frac{1}{x^2 + y^2} \, dA.
\]

Solution:
\[
\iint_D \frac{1}{x^2 + y^2} \, dA = \int_0^\pi \int_1^2 \frac{1}{r^2} r \, dr \, d\theta \\
= \left( \int_0^\pi d\theta \right) \left( \int_1^2 \frac{1}{r} \, dr \right) \\
= \pi \ln(r) \bigg|_{r=1}^{r=2} \\
= \pi (\ln(2) - \ln(1)) \\
\approx 2.18.
\]

5. Let \( D \) be the triangular lamina with vertices at \((0,0)\), \((1,1)\), and \((4,0)\) and let \( \rho \) be the density function \( \rho(x,y) = y \).

(a) Draw a picture of this lamina. In your picture, show equations for all three of the boundary curves.
(b) Set up and evaluate a double integral that gives the total mass of the lamina.

(c) Set up but do not evaluate double integrals that give the $x$ and $y$ coordinates of the center of mass of this lamina.

Solution: The lamina is pictured below.

The total mass of the lamina is

\[
m = \iint_D \rho(x, y) \, dA
\]

\[
= \int_0^1 \int_y^{3y+4} y \, dx \, dy
\]

\[
= \int_0^1 (xy)_{x=y}^{3y+4} \, dy
\]

\[
= \int_0^1 ((-3y + 4) y - y^2) \, dy
\]

\[
= \int_0^1 (-4y^2 + 4y) \, dy
\]

\[
= -4 \int_0^1 (y^2 - y) \, dy
\]

\[
= \frac{2}{3}.
\]
The $x$ coordinate of the center of mass is
\[ \bar{x} = \frac{1}{m} \iint_D x \rho(x, y) \, dA \]
\[ = \frac{3}{2} \int_0^1 \int_y^{3y+4} xy \, dx \, dy \]
\[ = \frac{3}{2}. \]

The $y$ coordinate of the center of mass is
\[ \bar{y} = \frac{1}{m} \iint_D y \rho(x, y) \, dA \]
\[ = \frac{3}{2} \int_0^1 \int_y^{3y+4} y^2 \, dx \, dy \]
\[ = \frac{1}{2}. \]

6. Draw a sketch of the two-dimensional vector field
\[ \mathbf{F}(x, y) = -yi + \frac{1}{2}j \]
with vectors based at the nine points indicated below.
Solution: