2.

\[ r(1) = (1,1) \]
\[ r(1.1) = (1.21, 1.1) \]
\[ r(1.1) - r(1) = (0.21, 0.1) \]
\[ \frac{r(1.1) - r(1)}{0.1} = (2.1, 1) \]
\[ r'(1) = (2, 1) \]

Since \( h = 0.1 \) is quite small (close to zero), we have

\[ \frac{r(1.1) - r(1)}{0.1} \approx r'(1). \]
3. (Numbers 3, 5, and 7 are different problems than the ones in the book.)
For \( \mathbf{r}(t) = (\cos(t), \sin(t)) \), we have \( \mathbf{r}'(t) = (-\sin(t), \cos(t)) \). Thus
\[
\mathbf{r}'(\pi/4) = (-\sin(\pi/4), \cos(\pi/4)) = \left(-\sqrt{2}/2, \sqrt{2}/2\right).
\]
Also,
\[
\mathbf{r}(\pi/4) = (\cos(\pi/4), \sin(\pi/4)) = \left(\sqrt{2}/2, \sqrt{2}/2\right).
\]

5. For \( \mathbf{r}(t) = (1 + t)\mathbf{i} + t^2\mathbf{j} \), we have \( \mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} \). Thus
\[
\mathbf{r}'(1) = \mathbf{i} + 2\mathbf{j}.
\]
Also,
\[
\mathbf{r}(1) = 2\mathbf{i} + \mathbf{j}.
\]
7. For \( \mathbf{r}(t) = e^{t}i + e^{-2t}j \), we have \( \mathbf{r}'(t) = e^{t}i - 2e^{-2t}j \). Thus

\[ \mathbf{r}'(0) = i - 2j. \]

Also,

\[ \mathbf{r}(0) = i + j. \]
9. For \( \mathbf{r}(t) = \langle t^2, 1 - t, \sqrt{t} \rangle \), we have
\[
\mathbf{r}'(t) = \left\langle 2t, -1, \frac{1}{2\sqrt{t}} \right\rangle.
\]
11. For \( \mathbf{r}(t) = e^t \mathbf{i} - \mathbf{j} + \ln(1 + 3t) \mathbf{k} \), we have
\[
\mathbf{r}'(t) = 2te^t \mathbf{i} + \frac{3}{1 + 3t} \mathbf{k}.
\]
13. For \( \mathbf{r}(t) = a + t \mathbf{b} + t^2 \mathbf{c} \), we have
\[
\mathbf{r}'(t) = \mathbf{b} + 2t \mathbf{c}.
\]
15. For \( \mathbf{r}(t) = \cos(t) \mathbf{i} + 3t \mathbf{j} + 2 \sin(2t) \mathbf{k} \), we have
\[
\mathbf{r}'(t) = -\sin(t) \mathbf{i} + 3 \mathbf{j} + 4 \cos(2t) \mathbf{k}.
\]
Thus \( \mathbf{r}'(0) = 3 \mathbf{j} + 4 \mathbf{k} \)
and
\[
\mathbf{T}(0) = \frac{1}{\lvert \mathbf{r}'(0) \rvert} \mathbf{r}'(0) = \frac{1}{5} (3 \mathbf{j} + 4 \mathbf{k}) = \frac{3}{5} \mathbf{j} + \frac{4}{5} \mathbf{k}.
\]
17. For \( \mathbf{r}(t) = \langle t, t^2, t^3 \rangle \), we have
\[
\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle
\]
\[
\mathbf{r}'(1) = \langle 1, 2, 3 \rangle
\]
\[
\lvert \mathbf{r}'(1) \rvert = \sqrt{14}
\]
\[
\mathbf{T}(1) = \frac{1}{\lvert \mathbf{r}'(1) \rvert} \mathbf{r}'(1) = \left\langle \frac{\sqrt{14}}{14}, \frac{2\sqrt{14}}{14}, \frac{3\sqrt{14}}{14} \right\rangle
\]
\[
\mathbf{r}''(t) = \langle 0, 2, 6t \rangle.
\]
Also
\[
\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 2t & 3t^2 \\
0 & 2 & 6t
\end{vmatrix}
= 6t^2 \mathbf{i} - 6t \mathbf{j} + 2 \mathbf{k}.
\]
19. The curve with parametric equations
\[
x = t^5
\]
\[
y = t^4
\]
\[
z = t^3
\]
has tangent vector $\langle 5t^4, 4t^3, 3t^2 \rangle$. At the point $(1, 1, 1)$ (which is on the curve), we have $t = 1$. Thus, the tangent vector at this point is $\langle 5, 4, 3 \rangle$. The tangent line at this point is the line with parametric equations
\begin{align*}
x &= 1 + 5t \\
y &= 1 + 4t \\
z &= 1 + 3t.
\end{align*}

21. The curve with parametric equations
\begin{align*}
x &= e^{-t} \cos(t) \\
y &= e^{-t} \sin(t) \\
z &= e^{-t}
\end{align*}
has tangent vector
\begin{align*}
e^{-t} \langle -\sin(t) - \cos(t), \cos(t) - \sin(t), -1 \rangle.
\end{align*}
At the point $(1, 0, 1)$ (which is on the curve), we have $t = 0$. Thus, the tangent vector at this point is $\langle -1, 1, -1 \rangle$. The tangent line at this point is the line with parametric equations
\begin{align*}
x &= 1 - t \\
y &= t \\
z &= 1 - t.
\end{align*}

23. The curve with parametric equations
\begin{align*}
x &= t \\
y &= \sqrt{2} \cos(t) \\
z &= \sqrt{2} \sin(t)
\end{align*}
has tangent vector
\begin{align*}
\langle 1, -\sqrt{2} \sin(t), \sqrt{2} \cos(t) \rangle.
\end{align*}
At the point $(\pi/4, 1, 1)$ (which is on the curve), we have $t = \pi/4$. Thus, the tangent vector at this point is $\langle 1, -1, 1 \rangle$. The tangent line at this point is the line with parametric equations
\begin{align*}
x &= \frac{\pi}{4} + t \\
y &= 1 - t \\
z &= 1 + t.
\end{align*}
25. (a) For \( \mathbf{r}(t) = \langle t^3, t^4, t^5 \rangle \), we have
\[
\mathbf{r}'(t) = \langle 3t^2, 4t^3, 5t^4 \rangle.
\]
The curve defined by \( \mathbf{r} \) is not smooth because \( \mathbf{r}'(0) = \mathbf{0} \).

(b) For \( \mathbf{r}(t) = \langle t^3 + t, t^4, t^5 \rangle \), we have
\[
\mathbf{r}'(t) = \langle 3t^2 + 1, 4t^3, 5t^4 \rangle.
\]
The curve defined by \( \mathbf{r} \) is smooth because there is no point on this curve at which \( \mathbf{r}'(0) = \mathbf{0} \).

(c) For \( \mathbf{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle \), we have
\[
\mathbf{r}'(t) = \langle -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t) \rangle.
\]
The curve defined by \( \mathbf{r} \) is not smooth because there are points on this curve at which \( \mathbf{r}'(0) = \mathbf{0} \). (See the graph below.)
Graph of $x = \cos^3(t)$, $y = \sin^3(t)$

27. The curve $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ has tangent vector $\mathbf{r}'_1(t) = \langle 1, 2t, 3t^2 \rangle$. At the origin, this tangent vector is $\mathbf{r}'_1(0) = \langle 1, 0, 0 \rangle$.

The curve $\mathbf{r}_2(t) = \langle \sin(t), \sin(2t), t \rangle$ has tangent vector $\mathbf{r}'_2(t) = \langle \cos(t), 2 \cos(2t), 1 \rangle$.

At the origin, this tangent vector is $\mathbf{r}'_2(0) = \langle 1, 2, 1 \rangle$.

The angle, $\theta$, between the two tangent vectors is given by

$$\cos(\theta) = \frac{\mathbf{r}'_1(0) \cdot \mathbf{r}'_2(0)}{|\mathbf{r}'_1(0)||\mathbf{r}'_2(0)|} = \frac{\sqrt{6}}{6}.$$

Thus

$$\theta = \arccos\left(\frac{\sqrt{6}}{6}\right) \approx 66^\circ.$$

29.

$$\int (16t^3i - 9t^2j + 25t^4k)\ dt = 4t^4i - 3t^3j + 5t^5k + C.$$

31. (different from problem 29 in book)

$$\int_{0}^{\pi/4} (\cos(2t)i + \sin(2t)j + t \sin(t)k)\ dt$$

$$= \left[ \left( \frac{1}{2} \sin(2t)i - \frac{1}{2} \cos(2t)j + (\sin(t) - t \cos(t))k \right) \right]_{t=0}^{t=\pi/4}$$

$$= \left( \frac{1}{2}i + \frac{\sqrt{2}}{2} - \frac{\pi}{4}, \frac{\sqrt{2}}{2} \right) - \left( -\frac{1}{2}j \right)$$

$$= \frac{1}{2}i + \frac{1}{2}j + \frac{\sqrt{2}}{2} \left( 4 - \frac{\pi}{4} \right)k.$$

33

$$\int (e^t i + 2tj + \ln(t)k)\ dt = e^t i + t^2j + (t \ln(t) - t)k + C.$$

35. If $\mathbf{r}'(t) = t^2i + 4t^3j-t^2k$, then $\mathbf{r}(t) = \frac{1}{6}t^3i + t^4j - \frac{1}{4}t^3k + C$ where $C$ is a constant vector. Since we must also have (according to the given information) that $\mathbf{r}(0) = j$, then we must have

$$\frac{1}{3}(0)^3i + (0)^4j - \frac{1}{3}(0)^3k + C = j.$$

This implies that $C = j$ and hence that

$$\mathbf{r}(t) = \frac{1}{3}t^3i + (t^4 + 1)j - \frac{1}{3}t^3k.$$
37. Let \( \mathbf{u}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle \) and let \( \mathbf{v}(t) = \langle g_1(t), g_2(t), g_3(t) \rangle \). Then
\[
\begin{align*}
\mathbf{u}(t) + \mathbf{v}(t) &= \langle f_1(t) + g_1(t), f_2(t) + g_2(t), f_3(t) + g_3(t) \rangle, \\
\mathbf{u}'(t) &= \langle f'_1(t), f'_2(t), f'_3(t) \rangle, \\
\mathbf{v}'(t) &= \langle g'_1(t), g'_2(t), g'_3(t) \rangle.
\end{align*}
\]
Thus
\[
\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \left( \frac{d}{dt}(f_1(t) + g_1(t)), \frac{d}{dt}(f_2(t) + g_2(t)), \frac{d}{dt}(f_3(t) + g_3(t)) \right)
\]
\[
= \langle f'_1(t) + g'_1(t), f'_2(t) + g'_2(t), f'_3(t) + g'_3(t) \rangle
\]
\[
= \langle f'_1(t), f'_2(t), f'_3(t) \rangle + \langle g'_1(t), g'_2(t), g'_3(t) \rangle
\]
\[
= \mathbf{u}'(t) + \mathbf{v}'(t).
\]

39. Let \( \mathbf{u}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle \) and let \( \mathbf{v}(t) = \langle g_1(t), g_2(t), g_3(t) \rangle \). To make notation less cumbersome, we will suppress the \( t \)s that occur everywhere.

Thus
\[
\begin{align*}
\mathbf{u}' &= \langle f'_1, f'_2, f'_3 \rangle, \\
\mathbf{v}'(t) &= \langle g'_1, g'_2, g'_3 \rangle,
\end{align*}
\]
and
\[
\mathbf{u}(t) \times \mathbf{v}(t) = \begin{vmatrix}
i & j & k \\
f_1 & f_2 & f_3 \\
g_1 & g_2 & g_3
\end{vmatrix}
\]
\[
= (f_2 g_3 - f_3 g_2) \mathbf{i} - (f_1 g_3 - f_3 g_1) \mathbf{j} + (f_1 g_2 - f_2 g_1) \mathbf{k}.
\]
Thus,
\[
\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = (f_2 g'_3 + f'_2 g_3 - f_3 g'_2 - f'_3 g_2) \mathbf{i} - (f_1 g'_3 + f'_1 g_3 - f_3 g'_1 - f'_3 g_1) \mathbf{j} + (f_1 g'_2 + f'_1 g_2 - f_2 g'_1 - f'_2 g_1) \mathbf{k}.
\]

Also,
\[
\mathbf{u}(t) \times \mathbf{v}'(t) = \begin{vmatrix}
i & j & k \\
f_1 & f_2 & f_3 \\
g_1 & g_2 & g_3
\end{vmatrix}
\]
\[
= (f_2 g'_3 - f_3 g'_2) \mathbf{i} - (f_1 g'_3 - f_3 g'_1) \mathbf{j} + (f_1 g'_2 - f_2 g'_1) \mathbf{k}.
\]
and

\[
\mathbf{u}'(t) \times \mathbf{v}(t) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\mathbf{f}'_1 & \mathbf{f}'_2 & \mathbf{f}'_3 \\
g_1 & g_2 & g_3
\end{vmatrix} = (f'_2g_3 - f'_3g_2)i - (f'_1g_3 - f'_3g_1)j + (f'_1g_2 - f'_2g_1)k
\]

so

\[
\mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t) = (f_2g'_3 - f_3g'_2 + f'_2g_3 - f'_3g_2)i
\]

\[
- (f'_1g_3 - f'_3g_1 + f'_1g_3 - f'_3g_1)j
\]

\[
+ (f'_1g_2 - f'_2g_1 + f'_1g_2 - f'_2g_1)k
\]

which shows that

\[
\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t).
\]

41. For \( \mathbf{u}(t) = \mathbf{i} - 2t^2\mathbf{j} + 3t^3\mathbf{k} \) and \( \mathbf{v}(t) = t\mathbf{i} + \cos(t)\mathbf{j} + \sin(t)\mathbf{k} \), we have

\[
\mathbf{u}'(t) = -4t\mathbf{j} + 9t^2\mathbf{k}
\]

\[
\mathbf{v}'(t) = \mathbf{i} - \sin(t)\mathbf{j} + \cos(t)\mathbf{k}.
\]

Thus

\[
\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)
\]

\[
= (1 + 2t^2 \sin(t) + 3t^3 \cos(t))
\]

\[
+ (-4t \cos(t) + 9t^2 \sin(t))
\]

\[
= 1 + 11t^2 \sin(t) + 3t^3 \cos(t) - 4t \cos(t).
\]

43.

\[
\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{r}'(t)) = \mathbf{r}(t) \times \mathbf{r}''(t) + \mathbf{r}'(t) \times \mathbf{r}'(t).
\]

However, recall that if \( \mathbf{a} \) is any vector, then \( \mathbf{a} \times \mathbf{a} = \mathbf{0} \). Thus \( \mathbf{r}'(t) \times \mathbf{r}'(t) = \mathbf{0} \) and therefore

\[
\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{r}'(t)) = \mathbf{r}(t) \times \mathbf{r}''(t).
\]