1. The curve \( \mathbf{r}(t) = (2\sin(t), 5t, 2\cos(t)) \), \(-10 \leq t \leq 10\) (which is a piece of a helix) has length

\[
\int_{-10}^{10} |\mathbf{r}'(t)| \, dt = \int_{-10}^{10} \sqrt{\left(2\cos(t)\right)^2 + (5)^2 + (-2\sin(t))^2} \, dt
\]

\[
= \int_{-10}^{10} \sqrt{29} \, dt
\]

\[
= 20\sqrt{29}.
\]

3. The curve \( \mathbf{r}(t) = \left(\sqrt{2} t, e^t, e^{-t}\right) \), \(0 \leq t \leq 1\) has length

\[
\int_{0}^{1} |\mathbf{r}'(t)| \, dt = \int_{0}^{1} \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (e^{-t})^2} \, dt
\]

\[
= \int_{0}^{1} \sqrt{e^t + e^{-t}} \, dt
\]

\[
= \int_{0}^{1} (e^t + e^{-t}) \, dt
\]

\[
= (e^t - e^{-t}) \bigg|_{t=0}^{t=1}
\]

\[
= (e - e^{-1}) - (e^0 - e^0)
\]

\[
= (e - e^{-1}) - (1 - 1)
\]

\[
= e - e^{-1}.
\]

7. For \( \mathbf{r}(t) = e^t \sin(t)\mathbf{i} + e^t \cos(t)\mathbf{j} \), we have

\[
\mathbf{r}'(t) = e^t (\cos(t) + \sin(t))\mathbf{i} + e^t (-\sin(t) + \cos(t))\mathbf{j}
\]

and

\[
|\mathbf{r}'(t)| = \sqrt{(e^t (\cos(t) + \sin(t)))^2 + (e^t (-\sin(t) + \cos(t)))^2}
\]

\[
= \sqrt{2e^{2t}}
\]

\[
= \sqrt{2} e^t.
\]

Thus, the arc length function starting from \( t = 0 \) and measured in the direction of increasing \( t \) is

\[
s = \int_{0}^{t} |\mathbf{r}'(u)| \, du
\]

\[
= \int_{0}^{t} \sqrt{2} e^u \, du
\]

\[
= \sqrt{2} (e^t - 1).
\]

Solving the equation \( \sqrt{2} (e^t - 1) = s \) for \( t \), we obtain
\[ t = \ln \left( 1 + \frac{1}{\sqrt{2}} s \right). \]

Thus, using arc length \( s \) as our parameter, we can write the equation of this curve as

\[
\mathbf{q}(s) = \left( 1 + \frac{1}{\sqrt{2}} s \right) \sin \left( \ln \left( 1 + \frac{1}{\sqrt{2}} s \right) \right) \mathbf{i} \\
+ \left( 1 + \frac{1}{\sqrt{2}} s \right) \cos \left( \ln \left( 1 + \frac{1}{\sqrt{2}} s \right) \right) \mathbf{j}.
\]

9. For \( \mathbf{r}(t) = 3 \sin(t) \mathbf{i} + 4t \mathbf{j} + 3 \cos(t) \mathbf{k} \), we have

\[ \mathbf{r}'(t) = 3 \cos(t) \mathbf{i} + 4 \mathbf{j} - 3 \sin(t) \mathbf{k} \]

and

\[ |\mathbf{r}'(t)| = \sqrt{(3 \cos(t))^2 + (4)^2 + (-3 \sin(t))^2} = 5 \]

Thus, the arc length function starting from \( t = 0 \) and measured in the direction of increasing \( t \) is

\[ s = \int_0^t |\mathbf{r}'(u)| \, du = \int_0^t 5 \, du = 5t. \]

Thus, using arc length \( s \) as our parameter, we can write the equation of this curve as

\[ \mathbf{q}(s) = 3 \sin \left( \frac{1}{5} s \right) \mathbf{i} + \frac{4}{5} s \mathbf{j} + 3 \cos \left( \frac{1}{5} s \right) \mathbf{k} \]

11. For \( \mathbf{r}(t) = \langle 2 \sin(t), 5t, 2 \cos(t) \rangle \), the tangent vector is

\[ \mathbf{r}'(t) = \langle 2 \cos(t), 5, -2 \sin(t) \rangle \]

and the length of this tangent vector is

\[ |\mathbf{r}'(t)| = \sqrt{(2 \cos(t))^2 + (5)^2 + (-2 \sin(t))^2} = \sqrt{29}. \]

The unit tangent vector is

\[ \mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|} \mathbf{r}'(t) = \frac{1}{\sqrt{29}} \langle 2 \cos(t), 5, -2 \sin(t) \rangle. \]

Note that

\[ \mathbf{T}'(t) = \frac{1}{\sqrt{29}} \langle -2 \sin(t), 0, -2 \cos(t) \rangle \]

and the length of this vector is

\[ |\mathbf{T}'(t)| = \frac{1}{\sqrt{29}} \sqrt{(-2 \sin(t))^2 + (0)^2 + (-2 \cos(t))^2} = \frac{2}{\sqrt{29}}. \]

Thus, the unit normal vector is
In order to find the curvature using the formula
\[ \kappa = \left| \frac{dT}{ds} \right|, \]
we must first find the relationship between the parameter \( t \) and the arc length \( s \). This is similar to the work done in problem 1 of this exercise set. We obtain
\[ s = \sqrt{29} t. \]
Now note (by the Chain Rule) that
\[ \frac{dT}{dt} = \frac{ds}{dt} \frac{dT}{ds} \]
which means that
\[ \frac{dT}{ds} = \frac{\frac{dT}{dt}}{\frac{ds}{dt}} \]
\[ = \frac{1}{\sqrt{29}} \left( \frac{1}{\sqrt{29}} (-2 \sin(t), 0, -2 \cos(t)) \right) \]
\[ = \frac{1}{29} (-2 \sin(t), 0, -2 \cos(t)) \]
\[ = -\frac{2}{29} (\sin(t), 0, \cos(t)). \]
We now obtain
\[ \kappa = \left| \frac{dT}{ds} \right| = \frac{2}{29}. \]
Since this curve is a helix, it makes sense that it has constant curvature. Also, since we did all of the work of computing \( |T'(t)| \) and \( |r'(t)| \), it is much easier to compute the curvature using the formula
\[ \kappa = \frac{|T'(t)|}{|r'(t)|} = \frac{2}{\sqrt{29}} = \frac{2}{29} \]
since that way we don’t have to deal with \( s \) at all.

13. For \( r(t) = \langle \frac{1}{4} t^3, t^2, 2t \rangle \), the tangent vector is
\[ r'(t) = \langle t^2, 2t, 2 \rangle \]
and the length of this tangent vector is
The unit tangent vector is
\[ T(t) = \frac{1}{|r'(t)|} r'(t) \]
\[ = \frac{1}{t^2 + 2} \langle t^2, 2t, 2 \rangle \]
\[ = \left\langle \frac{t^2}{t^2 + 2}, \frac{2t}{t^2 + 2}, \frac{2}{t^2 + 2} \right\rangle. \]

Note that
\[ T'(t) = \left\langle \frac{4t}{(t^2 + 2)^2}, \frac{4 - 2t^2}{(t^2 + 2)^2}, \frac{-4t}{(t^2 + 2)^2} \right\rangle \]
and the length of this vector is
\[ |T'(t)| = \sqrt{\left( \frac{4t}{(t^2 + 2)^2} \right)^2 + \left( \frac{4 - 2t^2}{(t^2 + 2)^2} \right)^2 + \left( \frac{-4t}{(t^2 + 2)^2} \right)^2} \]
\[ = \frac{1}{(t^2 + 2)^2} \sqrt{16t^2 + (16 - 16t^2 + 4t^4) + 16t^2} \]
\[ = \frac{1}{(t^2 + 2)^2} \sqrt{4t^4 + 16t^2 + 16} \]
\[ = \frac{2}{(t^2 + 2)^2} \sqrt{(t^2 + 2)^2} \]
\[ = \frac{2}{t^2 + 2}. \]

Thus, the unit normal vector is
\[ N(t) = \frac{1}{|T'(t)|} T'(t) \]
\[ = \frac{t^2 + 2}{2} \left\langle \frac{4t}{(t^2 + 2)^2}, \frac{4 - 2t^2}{(t^2 + 2)^2}, \frac{-4t}{(t^2 + 2)^2} \right\rangle \]
\[ = \left\langle \frac{2t}{t^2 + 2}, \frac{2 - t^2}{t^2 + 2}, \frac{-2t}{t^2 + 2} \right\rangle. \]

The curvature using the formula
\[ \kappa = \frac{|T'(t)|}{|r'(t)|} = \frac{2}{t^2 + 2} = \frac{2}{(t^2 + 2)^2}. \]

15. For \( r(t) = t^2 \mathbf{i} + t \mathbf{k} \) we have
\[ r'(t) = 2i + k \]
\[ |r'(t)| = \sqrt{(2t)^2 + 1^2} = \sqrt{4t^2 + 1} \]
\[ r''(t) = 2i \]
\[ r'(t) \times r''(t) = (2t_i + k) \times 2i \]
\[ = 4t(i \times i) + 2(k \times i) \]
\[ = 2j \]
\[ |r'(t) \times r''(t)| = 2. \]

Thus, the curvature is
\[ \kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{2}{(4t^2 + 1)^{3/2}}. \]

17. For \( r(t) = \sin(t)i + \cos(t)j + \sin(t)k \) we have
\[ r'(t) = \cos(t)i - \sin(t)j + \cos(t)k \]
\[ |r'(t)| = \sqrt{\cos^2(t) + (-\sin(t))^2 + \cos^2(t)} \]
\[ = \sqrt{1 + \cos^2(t)} \]
\[ r''(t) = -\sin(t)i - \cos(t)j - \sin(t)k \]
\[ r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\
\cos(t) & -\sin(t) & \cos(t) \\
-\sin(t) & -\cos(t) & -\sin(t) \end{vmatrix} \]
\[ = i - k. \]
\[ |r'(t) \times r''(t)| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}. \]

Thus, the curvature is
\[ \kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{\sqrt{2}}{(1 + \cos^2(t))^{3/2}}. \]

19. For \( r(t) = \sqrt{2}ti + e^tj + e^{-t}k \) we have
\[
\mathbf{r}'(t) = \sqrt{2} \mathbf{i} + e^t \mathbf{j} - e^{-t} \mathbf{k}
\]
\[
|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (e^{-t})^2}
\]
\[
= \sqrt{(e^t)^2 + 2e^t e^{-t} + (e^{-t})^2}
\]
\[
= \sqrt{(e^t + e^{-t})^2}
\]
\[
= e^t + e^{-t}
\]
\[
\mathbf{r}''(t) = e^t \mathbf{j} + e^{-t} \mathbf{k}
\]
\[
\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\sqrt{2} & e^t & -e^{-t} \\
0 & e^t & e^{-t}
\end{vmatrix}
\]
\[
= 2\mathbf{i} - \sqrt{2} e^{-t} \mathbf{j} + \sqrt{2} e^t \mathbf{k}.
\]
\[
|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{(2)^2 + \left(-\sqrt{2} e^{-t}\right)^2 + \left(\sqrt{2} e^t\right)^2}
\]
\[
= \sqrt{4 + 2(e^{-t})^2 + 2(e^t)^2}
\]
\[
= \sqrt{2} \sqrt{(e^t)^2 + 2e^t e^{-t} + (e^{-t})^2}
\]
\[
= \sqrt{2} \sqrt{(e^t + e^{-t})^2}
\]
\[
= \sqrt{2} (e^t + e^{-t}).
\]

Thus, the curvature is
\[
\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{2} (e^t + e^{-t})}{(e^t + e^{-t})^3} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}.
\]

Since the point \((0, 1, 1)\) on this curve corresponds to the parameter value \(t = 0\), we see that the curvature at this point is
\[
\kappa(0) = \frac{\sqrt{2}}{(e^0 + e^{-0})^2} = \frac{\sqrt{2}}{4}.
\]

21. For the curve \(f(x) = x^3\), we have
\[
f'(x) = 3x^2
\]
\[
f''(x) = 6x
\]
so the curvature is
\[
\kappa(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}} = \frac{6|x|}{\left(1 + 9x^4\right)^{3/2}}.
\]

23. For the curve \(f(x) = 4x^{5/2}\), we have
\[
f'(x) = 10x^{3/2}
\]
\[
f''(x) = 15x^{1/2}\]
so the curvature is
\[ \kappa(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}} = \frac{15x^{1/2}}{(1 + 100x^3)^{3/2}}. \]

25. For the curve \( f(x) = e^x \), we have
\[
\begin{align*}
f'(x) &= e^x \\
f''(x) &= e^x
\end{align*}
\]
so the curvature is
\[
\kappa(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}}.
\]

We would like to find at what value of \( x \) the curvature is maximum: Since
\[
\kappa'(x) = \frac{(1 + e^{2x})^{3/2}e^x - e^x \left( \frac{3}{2} (1 + e^{2x})^{1/2} (2e^{2x}) \right)}{(1 + e^{2x})^3}
\]
\[
= \frac{e^x(1 + e^{2x})^{1/2} (1 - 2e^{2x})}{(1 + e^{2x})^3}
\]
\[
= \frac{e^x (1 - 2e^{2x})}{(1 + e^{2x})^{5/2}}
\]
we see that \( \kappa'(x) = 0 \) when \( 1 - 2e^{2x} = 0 \). Solving this equation for \( x \) gives us
\[
e^{2x} = 1/2 \text{ or } (e^x)^2 = 1/2 \text{ or } e^x = \sqrt{2}/2 \text{ or } x = \ln \left( \frac{\sqrt{2}}{2} \right).
\]
Since this \( x \) gives us the only critical point of \( \kappa \) and since it is clear that the curvature must be maximum at some point for the graph of \( f(x) = e^x \), we conclude that the curvature is maximum at the point
\[
\left( \ln \left( \frac{\sqrt{2}}{2} \right), \frac{\sqrt{2}}{2} \right) \approx (-0.34637, 0.70711).
\]
27. The curvature of the curve (drawn on page 724 of the textbook) seems to be greater at point $P$.

35. For $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle$, we have

$$\mathbf{r}'(t) = \langle 2t, 2t^2, 1 \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{4t^2 + 4t^4 + 1}$$

$$= 2 \sqrt{t^4 + t^2 + \frac{1}{4}}$$

$$= 2 \sqrt{(t^2 + \frac{1}{2})^2}$$

$$= 2 \left( t^2 + \frac{1}{2} \right)$$

$$= 2t^2 + 1$$

$$\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|} \mathbf{r}'(t)$$

$$= \left\langle \frac{2t}{2t^2 + 1}, \frac{2t^2}{2t^2 + 1}, \frac{1}{2t^2 + 1} \right\rangle$$

$$\mathbf{T}'(t) = \left\langle \frac{2 - 4t^2}{(2t^2 + 1)^2}, \frac{4t}{(2t^2 + 1)^3}, \frac{-4t}{(2t^2 + 1)^2} \right\rangle$$

The point $\left( 1, \frac{2}{3}, 1 \right)$ corresponds to the parameter value $t = 1$ so
\[ \mathbf{T}(1) = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle \]
\[ \mathbf{N}(1) = \frac{1}{|\mathbf{T}'(1)|} \mathbf{T}'(1) \]
\[ = \frac{1}{\sqrt{\left(\frac{2}{9}\right)^2 + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^2}} \left\langle -\frac{2}{9}, \frac{4}{9}, -\frac{4}{9} \right\rangle \]
\[ = \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle \]

and

\[ \mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) \]
\[ = \left| \begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\
\end{array} \right| \]
\[ = -\frac{2}{3} \mathbf{i} + \frac{1}{3} \mathbf{j} + \frac{2}{3} \mathbf{k}. \]

37. For \( \mathbf{r}(t) = (2 \sin(3t), t, 2 \cos(3t)) \), we have
\[ \mathbf{r}'(t) = (6 \cos(3t), 1, -6 \sin(3t)) \]
\[ |\mathbf{r}'(t)| = \sqrt{37} \]
\[ \mathbf{T}(t) = \frac{1}{\sqrt{37}} \left\langle 6 \cos(3t), 1, -6 \sin(3t) \right\rangle \]
\[ \mathbf{T}'(t) = \frac{1}{\sqrt{37}} \left\langle -18 \sin(3t), 0, -18 \cos(3t) \right\rangle \]
The point \((0, \pi, -2)\) corresponds to the parameter value \( t = \pi \) so
\[ \mathbf{T}(\pi) = \frac{1}{\sqrt{37}} \left\langle -6, 1, 0 \right\rangle \]
\[ \mathbf{N}(\pi) = \frac{1}{|\mathbf{T}'(\pi)|} \mathbf{T}'(\pi) \]
\[ = \left\langle 0, 0, 1 \right\rangle \]

and

\[ \mathbf{B}(\pi) = \mathbf{T}(\pi) \times \mathbf{N}(\pi) \]
\[ = \left| \begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-\frac{6}{\sqrt{37}} & \frac{1}{\sqrt{37}} & 0 \\
0 & 0 & 1 \\
\end{array} \right| \]
\[ = \frac{1}{\sqrt{37}} \mathbf{i} + \frac{6}{\sqrt{37}} \mathbf{j}. \]

Since the binormal vector is orthogonal to the osculating plane, the osculating plane has equation
\[ x + 6(y - \pi) = 0 \]

and since the unit tangent vector is orthogonal to the normal plane, the normal plane has equation
\[ -6x + (y - \pi) = 0. \]

39. The ellipse \(9x^2 + 4y^2 = 36\) can be written as
\[ \frac{x^2}{4} + \frac{y^2}{9} = 1. \]

Parametric equations of this ellipse are
\[ x = 2 \cos(t) \]
\[ y = 3 \sin(t). \]

For this parametric curve \( \mathbf{r}(t) \) we have
\[ \mathbf{r}'(t) = \langle -2 \sin(t), 3 \cos(t) \rangle \]
\[ |\mathbf{r}'(t)| = \sqrt{4 \sin^2(t) + 9 \cos^2(t)} \]
\[ \mathbf{r}''(t) = \langle -2 \cos(t), -3 \sin(t) \rangle \]
\[ \mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 \sin(t) & 3 \cos(t) & 0 \\ -2 \cos(t) & -3 \sin(t) & 0 \end{vmatrix} = 6 \mathbf{k} \]
\[ |\mathbf{r}'(t) \times \mathbf{r}''(t)| = 6 \]
\[ \kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{6}{\left(4 \sin^2(t) + 9 \cos^2(t)\right)^{3/2}}. \]

The point \((2, 0)\) on the ellipse corresponds to the parameter value \(t = 0\) so at this point the curvature is
\[ \kappa = \frac{6}{\left(4 \sin^2(0) + 9 \cos^2(0)\right)^{3/2}} = \frac{2}{3} \]

The osculating circle at this point thus has radius \(9/2\) and is centered at the point \((-\frac{5}{2}, 0)\). An equation for the osculating circle is
\[ \left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4}. \]

Parametric equations of this circle (used to graph the circle) are
\[ x = -\frac{5}{2} + \frac{9}{2} \cos(t) \]
\[ y = \frac{9}{2} \sin(t). \]

The point \((0, 3)\) on the ellipse corresponds to the parameter value \(t = \pi/2\) so at this point the curvature is
\[ \kappa = \frac{6}{\left(4 \sin^2\left(\frac{\pi}{2}\right) + 9 \cos^2\left(\frac{\pi}{2}\right)\right)^{3/2}} = \frac{3}{4} \]

The osculating circle at this point thus has radius \(4/3\) and is centered at the
point \((0, \frac{5}{3})\). An equation for the osculating circle is
\[ x^2 + \left( y - \frac{5}{3} \right)^2 = \frac{16}{9}. \]
Parametric equations of this circle (used to graph the circle) are
\[ x = \frac{4}{3} \cos(t) \]
\[ y = \frac{5}{3} + \frac{4}{3} \sin(t). \]