Answers and Solutions to Section 11.2 Homework
Problems 1-17 (odd) and 25-35 (odd)

1. Just knowing that \( \lim_{(x,y) \to (3,1)} f(x,y) = 6 \) does not tell us anything about the value of \( f(3,1) \). However, if we know that \( f \) is a continuous function (or just that \( f \) is continuous at the point \((3,1)\)), then the fact that \( \lim_{(x,y) \to (3,1)} f(x,y) = 6 \) tells us that \( f(3,1) = 6 \).

3. Since \( f(x,y) = x^2y + x^3y^2 - 5 \) is a rational function, it is continuous at all points in its domain. Thus
\[
\lim_{(x,y) \to (0,0)} f(x,y) = f(0,0) = -\frac{5}{2}.
\]

5. \[
\lim_{(x,y) \to (5,-2)} \left( x^5 + 4x^3y - 5xy^2 \right) = (5)^5 + 4(5)^3(-2) - 5(5)(-2)^2 = 2,025.
\]

7. The limit \( \lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} \) does not exist.

If we let \((x, y)\) approach \((0,0)\) along the \( x \) axis (where \( y = 0 \)), then we obtain
\[
\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2}{x^2} = \lim_{x \to 0} 1 = 1.
\]

If we let \((x, y)\) approach \((0,0)\) along the \( y \) axis (where \( x = 0 \)), then we obtain
\[
\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2} = \lim_{y \to 0} \frac{0}{0 + y^2} = \lim_{y \to 0} 0 = 0.
\]
If we let \((x, y)\) approach \((0, 0)\) along the line \(y = x\), then we obtain
\[
\lim_{{(x, y) \to (0, 0) \text{ with } y=x}} \frac{8x^2y^2}{x^4 + y^4} = \lim_{{x \to 0}} \frac{8x^2x^4}{x^4 + x^4} = \lim_{{x \to 0}} 4 = 4.
\]

11. The limit
\[
\lim_{{(x, y) \to (0, 0)}} \frac{xy}{\sqrt{x^2 + y^2}} = 0.
\]
Note that for all points \((x, y)\) such that \((x, y) \neq (0, 0)\), we have
\[
\left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = \frac{|x||y|}{\sqrt{x^2 + y^2}} = \frac{\sqrt{x^2}|y|}{\sqrt{x^2 + y^2}} = \frac{x^2}{x^2 + y^2}|y|.
\]
Since
\[
0 \leq \frac{x^2}{x^2 + y^2} \leq 1
\]
for all \((x, y) \neq (0, 0)\), we have
\[
\left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = \sqrt{\frac{x^2}{x^2 + y^2}}|y| \leq |y|
\]
for all \((x, y) \neq (0, 0)\). Thus (using the fact that the condition \(|A| \leq B\) is equivalent to the condition \(−B \leq A \leq B\)), we obtain
\[
-|y| \leq \frac{xy}{\sqrt{x^2 + y^2}} \leq |y|
\]
for all \((x, y) \neq (0, 0)\).
Since \(\lim_{{(x, y) \to (0, 0)}} (−|y|) = 0\) and \(\lim_{{(x, y) \to (0, 0)}} |y| = 0\), then the Squeeze Theorem tells us that
\[
\lim_{{(x, y) \to (0, 0)}} \frac{xy}{\sqrt{x^2 + y^2}} = 0.
\]

13. The limit
\[
\lim_{{(x, y) \to (0, 0)}} \frac{2x^2y}{x^4 + y^2}
\]
does not exist.
If we let \((x, y)\) approach \((0, 0)\) along the \(x\) axis (where \(y = 0\)), then we obtain
\[
\lim_{{(x, y) \to (0, 0) \text{ with } y=0}} \frac{2x^2y}{x^4 + y^2} = \lim_{{x \to 0}} \frac{0}{x^4} = \lim_{{x \to 0}} 0 = 0.
\]
If we let \((x, y)\) approach \((0, 0)\) along the parabola \(y = x^2\), then we obtain
\[
\lim_{{(x, y) \to (0, 0) \text{ with } y=x^2}} \frac{2x^2y}{x^4 + y^2} = \lim_{{x \to 0}} \frac{2x^2x^2}{x^4 + (x^2)^2} = \lim_{{x \to 0}} 2 = 2.
\]
15. Observe that for all \((x, y) \neq (0, 0)\), we have

\[
\frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} = \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} \left(\frac{\sqrt{x^2 + y^2 + 1} + 1}{\sqrt{x^2 + y^2 + 1} + 1}\right) = \frac{(x^2 + y^2)}{\left(\sqrt{x^2 + y^2 + 1}\right)^2 - 1^2} = \frac{x^2 + y^2}{x^2 + y^2 + 1} + 1.
\]

Thus

\[
\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} = \lim_{(x,y)\to(0,0)} \left(\frac{\sqrt{x^2 + y^2 + 1} + 1}{\sqrt{x^2 + y^2 + 1} + 1}\right) = 2.
\]

17. The limit

\[
\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4}
\]
does not exist.

If we let \((x, y, z) \to (0, 0, 0)\) along the \(z\) axis (where \(x = y = 0\)), we obtain

\[
\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4} = \lim_{z\to0} \frac{0}{z^4} = 0.
\]

If we let \((x, y, z) \to (0, 0, 0)\) along the parabola where \(y = 0\) and \(x = z^2\), we obtain

\[
\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2 + y^2 + z^4} = \lim_{z\to0} \frac{z^2}{(z^2)^2 + z^4} = \frac{1}{2}.
\]

25. The function

\[
F(x, y) = \frac{1}{x^2 - y}
\]
is continuous throughout its domain which is all of \(\mathbb{R}^2\) except for the points that lie on the parabola \(y = x^2\).

27. Since that arctangent function is continuous everywhere (for all real numbers), the function \(F(x, y) = \arctan(x + \sqrt{y})\) is continuous everywhere that it is defined. It is defined at all points \((x, y)\) such that \(y \geq 0\).

29. The function

\[
F(x, y, z) = \frac{xyz}{x^2 + y^2 - z}
\]
is continuous at all points in its domain. Its domain it all of \(\mathbb{R}^3\) except for the points that lie on the paraboloid \(z = x^2 + y^2\).
31. The function
\[ f(x, y) = \begin{cases} \frac{x^2 y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases} \]
is continuous at all points \((x, y) \neq (0, 0)\) because it is a rational function that is defined at all such points. The only point in question is \((x, y) = (0, 0)\). It can be seen that \(f\) is not continuous at this point because
\[ \lim_{y \to 0} f(x, y) = \lim_{x \to 0} \frac{x^2 (0)^3}{2x^2 + (0)^2} = 0. \]

33. \[ \lim_{(x, y) \to (0, 0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r \to 0^+} \frac{(r \cos(\theta))^3 + (r \sin(\theta))^3}{(r \cos(\theta))^2 + (r \sin(\theta))^2} \]
\[ = \lim_{r \to 0^+} \frac{r^3 \cos^3(\theta) + r^3 \sin^3(\theta)}{r^2} \]
\[ = \lim_{r \to 0^+} r \left( \cos^3(\theta) + \sin^3(\theta) \right). \]

Since \(-2 \leq \cos^3(\theta) + \sin^3(\theta) \leq 2\) for all \(\theta\), we see that
\[-2r \leq r \left( \cos^3(\theta) + \sin^3(\theta) \right) \leq 2r \]
for all \(r > 0\). Since \(\lim_{r \to 0^+} (-2r) = 0\) and \(\lim_{r \to 0^+} (2r) = 0\), then the Squeeze Theorem tells us that
\[ \lim_{r \to 0^+} r \left( \cos^3(\theta) + \sin^3(\theta) \right) = 0 \]
and hence that
\[ \lim_{(x, y) \to (0, 0)} \frac{x^3 + y^3}{x^2 + y^2} = 0. \]

35. \[ \lim_{(x, y, z) \to (0, 0, 0)} \frac{xyz}{x^2 + y^2 + z^2} = \lim_{\rho \to 0^+} \frac{(\rho \sin(\phi) \cos(\theta)) (\rho \sin(\phi) \sin(\theta)) (\rho \cos(\phi))}{\rho^2} \]
\[ = \lim_{\rho \to 0^+} \left( \rho \sin^2(\phi) \cos(\phi) \sin(\theta) \cos(\theta) \right). \]

Since \(-1 \leq \sin^2(\phi) \cos(\phi) \sin(\theta) \cos(\theta) \leq 1\) for all \(\phi\) and \(\theta\), we can use the Squeeze Theorem (similar to problem 33) to conclude that
\[ \lim_{\rho \to 0^+} \left( \rho \sin^2(\phi) \cos(\phi) \sin(\theta) \cos(\theta) \right) = 0 \]
and hence that
\[ \lim_{(x, y, z) \to (0, 0, 0)} \frac{xyz}{x^2 + y^2 + z^2} = 0. \]