1. (a) $\partial T/\partial x$ is the rate of change of temperature with respect to longitude at a fixed latitude and fixed time. $\partial T/\partial y$ is the rate of change of temperature with respect to latitude at a fixed longitude and fixed time. $\partial T/\partial t$ is the rate of change of temperature with respect to time at a fixed latitude and fixed longitude (i.e., at a fixed location on the Earth).

(b) $f_x(158, 21, 9) > 0$ because at 9 a.m. the temperature is decreasing as we travel from west to east (which corresponds to decreasing longitude) along the line 21° N latitude. $f_y(158, 21, 9) < 0$ because at 9 a.m. the temperature is decreasing as we travel from south to north (which corresponds to increasing latitude) along the line 158° W longitude. $f_t(158, 21, 9) > 0$ because at 9 a.m. the temperature in Honolulu is increasing.

3. (a) We can estimate $f_T(-15, 30)$ to be the average of

$$\frac{f(-10, 30) - f(-15, 30)}{-10 - (-15)} = \frac{-20 - (-26)}{5} = 1.2^\circ C/\circ C$$

and

$$\frac{f(-20, 30) - f(-15, 30)}{-20 - (-15)} = \frac{-33 - (-26)}{-5} = 1.4^\circ C/\circ C.$$ 

Thus we estimate that

$$f_T(-15, 30) \approx 1.3^\circ C/\circ C.$$ 

This means that when the wind speed is 30 km/hr (constant), then the instantaneous rate of change of the wind chill index is about $1.3^\circ C$ per $1^\circ C$ increase in actual temperature when the temperature is $-15^\circ C$.

We can estimate $f_v(-15, 30)$ to be the average of

$$\frac{f(-15, 40) - f(-15, 30)}{40 - 30} = \frac{-27 - (-26)}{10} = -0.1^\circ C/(\text{km/hr})$$
and
\[
\frac{f(-15, 20) - f(-15, 30)}{20 - 30} = \frac{-24 - (-26)}{-10} = -0.2^\circ C/ (\text{km/hr}).
\]
Thus we estimate that
\[
f_v (12, 20) \approx -0.15^\circ C/ (\text{km/hr}).
\]
This means that when the temperature is \(-15^\circ C\) (constant), then the instantaneous rate of change of the wind chill index is about \(-0.3^\circ C\) per 1 km/hr increase in wind speed when the temperature is \(-15^\circ C\).

(b) In general \(\partial W/\partial T > 0\) and \(\partial W/\partial v < 0\).

(c) It appears that
\[
\lim_{v \to \infty} \frac{\partial W}{\partial v} = 0.
\]

5. Assuming that the point \((1, 2)\) corresponds to the blue dot on the graph (shown in the book):

(a) \(f_x (1, 2) > 0\)

(b) \(f_y (1, 2) < 0\).

7. Graph \(c\) is the graph of \(f\). Graph \(b\) is the graph of \(f_x\). Graph \(a\) is the graph of \(f_y\).

9. For \(f(x, y) = 16 - 4x^2 - y^2\), we have
\[
f_x (x, y) = -8x
\]
\[
f_y (x, y) = -2y
\]
so
\[
f_x (1, 2) = -8
\]
\[
f_y (1, 2) = -4.
\]
This means that the slope of the tangent line to the curve of intersection of the surface \(z = 16 - 4x^2 - y^2\) and the plane \(x = 1\) at the point \((1, 2)\) on that curve is \(f_y (1, 2) = -4\).

Also, the slope of the tangent line to the curve of intersection of the surface \(z = 16 - 4x^2 - y^2\) and the plane \(y = 2\) at the point \((1, 2)\) on that curve is \(f_x (1, 2) = -8\).
78. The intersection of the surface \( z = 6 - x - x^2 - 2y^2 \) with the plane \( x = 1 \) is the parabola \( z = 4 - 2y^2 \). Since \( z_y = -4y \), the tangent line to this parabola at the point \((1, 2, -4)\) has direction vector \( \langle 0, 1, -8 \rangle \) and the parametric equations for this tangent line are

\[
\begin{align*}
    x &= 1 \\
    y &= 2 + t \\
    z &= -4 - 8t.
\end{align*}
\]

Graphs of the surface, the parabola, and the tangent line at the indicated point are shown below.
In a study of frost penetration, it was found that the temperature, $T$, at time $t$ (measured in days) at a depth $x$ (measured in feet) can be modelled by the function

$$T(x, t) = T_0 + T_1 e^{-\lambda x} \sin (\omega t - \lambda x)$$

where $\omega = 2\pi/365$ and $\lambda$ is a positive constant. ($T_0$ and $T_1$ are also constants.)

(a)

$$\frac{\partial T}{\partial x} = -T_1 \lambda e^{-\lambda x} \cos (\omega t - \lambda x) - T_1 \lambda e^{-\lambda x} \sin (\omega t - \lambda x)$$

$$= -T_1 \lambda e^{-\lambda x} (\cos (\omega t - \lambda x) + \sin (\omega t - \lambda x))$$

is the rate of change of ground temperature as depth increases at a fixed moment of time.

(b)

$$\frac{\partial T}{\partial t} = T_1 \omega e^{-\lambda x} \cos (\omega t - \lambda x)$$

is the rate of change of ground temperature with respect to time at a fixed depth.
(c) Note that
\[
T_{xx} = \frac{\partial}{\partial x} \left(-T_1 \lambda e^{-\lambda x} \left(\cos(\omega t - \lambda x) + \sin(\omega t - \lambda x)\right)\right)
\]
\[
= -T_1 \lambda e^{-\lambda x} \left(\lambda \sin(\omega t - \lambda x) - \lambda \cos(\omega t - \lambda x)\right)
\]
\[
+ T_1 \lambda^2 e^{-\lambda x} \left(\cos(\omega t - \lambda x) + \sin(\omega t - \lambda x)\right)
\]
\[
= 2T_1 \lambda^2 e^{-\lambda x} \cos(\omega t - \lambda x)
\]
This means that \( T_t = k T_{xx} \) for \( k = \frac{\omega}{2\lambda \omega^2} \).

83. The function under consideration is
\[
f(x, y) = \begin{cases} 
\frac{x^3 y - xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases}
\]
We will show that \( f_{xy}(0, 0) \) and \( f_{yx}(0, 0) \) both exist but that \( f_{xy}(0, 0) \neq f_{yx}(0, 0) \). This is not a contradiction of Clairaut’s Theorem because \( f_{xy} \) and \( f_{yx} \) are not continuous at the point \((0, 0)\). (Recall that Clairaut’s Theorem guarantees equality of \( f_{xy}(a, b) \) and \( f_{yx}(a, b) \) only if the functions \( f_{xy} \) and \( f_{yx} \) are continuous at all points in some neighborhood of \((a, b)\) —including at the point \((a, b)\) itself).

First, at any point \((x, y) \neq (0, 0)\), we have
\[
f_x(x, y) = \frac{\partial}{\partial x} \left(\frac{x^3 y - xy^3}{x^2 + y^2}\right)
\]
\[
= \frac{(x^2 + y^2) (3x^2 y - y^3) - (x^3 y - xy^3) (2x)}{(x^2 + y^2)^2}
\]
\[
= \frac{x^4 y - y^5 + 4x^2 y^3}{(x^2 + y^2)^2}.
\]
By similar reasoning
\[
f_y(x, y) = \frac{-xy^4 + x^5 - 4x^3 y^2}{(x^2 + y^2)^2}.
\]
In addition

\[ f_x (0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h^3(0) - h(0)^3}{h^2 + (0)^2} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} (0) = 0 \]

and by similar reasoning

\[ f_y (0, 0) = 0. \]

Therefore

\[ f_x (x, y) = \begin{cases} \frac{x^4y - y^5 + 4x^2y^3}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]

and

\[ f_y (x, y) = \begin{cases} \frac{-xy^4 + x^5 - 4x^3y^2}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}. \]

We now compute \( f_{xy} (0, 0) \) and \( f_{yx} (0, 0) \):

\[ f_{xy} (0, 0) = \lim_{h \to 0} \frac{f_x (0, h) - f_x (0, 0)}{h} = \lim_{h \to 0} \frac{(0)^4h - h^5 + 4(0)^2h^3}{(0^2 + h^2)^2} - 0 = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} \frac{-h}{h} = -1. \]

and

\[ f_{yx} (0, 0) = \lim_{h \to 0} \frac{f_y (h, 0) - f_y (0, 0)}{h} = \lim_{h \to 0} \frac{-h(0)^4 + h^5 - 4h(0)^3}{(h^2 + (0)^2)^2} - 0 = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} \frac{h}{h} = 1. \]
Observe that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

It is a tedious exercise to compute the general formulas for $f_{xy}(x, y)$ and $f_{yx}(x, y)$ so we will not do it!