Answers and Solutions to Section 11.4 Homework
Problems 1, 3, 9, 11, 13, 15, 23, 33, 35 and 37.
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1. For \( z = 4x^2 - y^2 + 2y \), we have

\[
\begin{align*}
    f_x &= 8x \\
    f_y &= -2y + 2
\end{align*}
\]

so

\[
\begin{align*}
    f_x (-1, 2) &= -8 \\
    f_y (-1, 2) &= -2
\end{align*}
\]

and the equation of the tangent plane at this point is

\[
z - 4 = -8 (x + 1) - 2(y - 2)
\]

or

\[
z = -8x - 2y.
\]

3. For \( z = y \cos (x - y) \), we have

\[
\begin{align*}
    f_x &= -y \sin (x - y) \\
    f_y &= y \sin (x - y) + \cos (x - y)
\end{align*}
\]

so

\[
\begin{align*}
    f_x (2, 2) &= 0 \\
    f_y (2, 2) &= 1
\end{align*}
\]

and the equation of the tangent plane at this point is

\[
z - 2 = 0 (x - 2) + 1 (y - 2)
\]

or

\[
z = y.
\]

9. For \( f(x, y) = x \sqrt{y} \), we have

\[
\begin{align*}
    f_x (x, y) &= \sqrt{y} \\
    f_y (x, y) &= x \cdot \frac{1}{2 \sqrt{y}} = \frac{x}{2 \sqrt{y}}.
\end{align*}
\]
Both of these partial derivatives are continuous at all points \((x, y) \in \mathbb{R}^2\) such that \(y > 0\). Since \((1, 4)\) is such a point, then \(f_x\) and \(f_y\) are continuous at the point \((1, 4)\) and, therefore, \(f\) is differentiable at \((1, 4)\). The linearization of \(f\) at the point \((1, 4)\) is

\[
L(x, y) = f(1, 4) + f_x(1, 4)(x - 1) + f_y(1, 4)(y - 4)
= 2 + 2(x - 1) + \frac{1}{4}(y - 4)
\]

or

\[
L(x, y) = 2x + \frac{1}{4}y - 1.
\]

11. For \(f(x, y) = \arctan(x + 2y)\), we have

\[
f_x(x, y) = \frac{1}{1 + (x + 2y)^2}
\]

\[
f_y(x, y) = \frac{2}{1 + (x + 2y)^2}.
\]

Both of these partial derivatives are continuous at all points \((x, y) \in \mathbb{R}^2\). Since \(f_x\) and \(f_y\) are continuous at the point \((1, 0)\), then \(f\) is differentiable at \((1, 0)\). The linearization of \(f\) at the point \((1, 0)\) is

\[
L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)
= \frac{\pi}{4} + \frac{1}{2}(x - 1) + \frac{1}{2}(y - 0)
\]

or

\[
L(x, y) = \frac{1}{2}x + y + \frac{\pi}{4} - \frac{1}{2}.
\]

13. For \(f(x, y) = \sqrt{20 - x^2 - 7y^2}\), we have

\[
f_x = \frac{-x}{\sqrt{20 - x^2 - 7y^2}}
\]

\[
f_y = \frac{-7y}{\sqrt{20 - x^2 - 7y^2}}
\]

so the linear approximation of \(f\) at the point \((2, 1)\) is

\[
L(x, y) = 3 - \frac{2}{3}(x - 2) - \frac{7}{3}(y - 1)
\]

or

\[
L(x, y) = -\frac{2}{3}x - \frac{7}{3}y + \frac{20}{3}.
\]
Since the point (1.95, 1.08) is close to the point (2, 1), then according to theory, we should have \( f(1.95, 1.08) \approx L(1.95, 1.08) \). Note that

\[
L(1.95, 1.08) = - \frac{2}{3} (1.95) - \frac{7}{3} (1.08) + \frac{20}{3} = \frac{2}{3} \left( \frac{39}{20} \right) - \frac{7}{3} \left( \frac{27}{25} \right) + \frac{20}{3} = -\frac{10}{150} = -\frac{195}{250} + \frac{378}{250} + \frac{1000}{250} = \frac{1370}{250} = \frac{195}{25} + \frac{378}{25} + \frac{1000}{25} = \frac{2378}{25} = 2.846.
\]

We conclude that

\[
\sqrt{20 - (1.95)^2 - 7(1.08)^2} = \sqrt{8.0327} \approx \frac{427}{150} = 2.846.
\]

The value of \( \sqrt{8.0327} \) calculated on the TI-83 calculator is 2.834201828.

15. For \( f(x, y, z) = w = \sqrt{x^2 + y^2 + z^2} \), we have

\[
\begin{align*}
f_x &= \frac{x}{w} \\
f_y &= \frac{y}{w} \\
f_z &= \frac{z}{w}
\end{align*}
\]

so the best linear approximation of \( f \) at the point (3, 2, 6) is

\[
L(x, y, z) = 7 + \frac{3}{7} (x - 3) + \frac{2}{7} (y - 2) + \frac{6}{7} (z - 6).
\]

Note that

\[
L(3.02, 1.97, 5.99) = 7 + \frac{3}{7} (.02) + \frac{2}{7} (-.03) + \frac{6}{7} (-.01)
= 7 + \frac{3}{7} \cdot \frac{350}{350} - \frac{3}{7} \cdot \frac{350}{350}
= \frac{2447}{350}
= 6.991428571.
\]

We thus estimate that

\[
\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = \sqrt{48.8814}
\approx 6.991428571.
\]

The value of \( \sqrt{48.8814} \) computed on the TI-83 calculator is 6.991523439.
23. For \( z = 5x^2 + y^2 \), we have
\[
dz = f_x \, dx + f_y \, dy = 10x \, dx + 2y \, dy.
\]
For \((x, y) = (1, 2)\) and \((x + dx, y + dy) = (1.05, 2.1)\), we have
\[
dz = 10 \times (1.05) + 2 \times (2.1) = 0.9.
\]
We thus estimate that \( \Delta z \approx 0.9 \). The actual value of \( \Delta z \) is
\[
\Delta z = f(1.05, 2.1) - f(1, 2) = \left( 5 \times (1.05)^2 + (2.1)^2 \right) - \left( 5 \times (1)^2 + 2^2 \right) = 9.9225 - 9 = 0.9225.
\]

25. The area of a rectangle with dimensions \( x \times y \) is given by the function \( A = xy \). The differential is
\[
dA = f_x \, dx + f_y \, dy = y \, dx + x \, dy.
\]
Setting \( dx = dy = 0.1 \), \( x = 30 \), and \( y = 24 \), we obtain
\[
dA = (24) \times (0.1) + (30) \times (0.1) = 5.4.
\]
The maximum area in the calculated area is thus 5.4 cm.

33. For the parametric surface
\[
r(u, v) = (u + v) \mathbf{i} + 3u^2 \mathbf{j} + (u - v) \mathbf{k}
\]
the point \((2, 3, 0)\) corresponds to the parameter values \((u, v) = (1, 1)\).
Since
\[
\begin{align*}
\mathbf{r}_u &= \mathbf{i} + 6v \mathbf{j} + \mathbf{k} \\
\mathbf{r}_v &= \mathbf{i} - \mathbf{k} \\
\mathbf{r}_u \times \mathbf{r}_v &= -6u \mathbf{i} + 2 \mathbf{j} - 6u \mathbf{k}
\end{align*}
\]
we see that
\[
\mathbf{r}_u \times \mathbf{r}_v (1, 1) = -6i + 2j - 6k
\]
so a normal vector to the tangent plane at this point on the surface is \(-3i + j - 3k\). An equation for the tangent plane is thus
\[
-3(x - 2) + 1(y - 3) - 3(z - 0) = 0
\]
or
\[
-3x + y - 3z + 3 = 0.
\]
35. For the parametric surface
\[ \mathbf{r}(u, v) = u^2\mathbf{i} + 2u \sin(v)\mathbf{j} + u \cos(v)\mathbf{k} \]
the point \((1, 0, 1)\) corresponds to the parameter values \((u, v) = (1, 0)\). Since
\[
\begin{align*}
\mathbf{r}_u &= 2u\mathbf{i} + 2 \sin(v)\mathbf{j} + \cos(v)\mathbf{k} \\
\mathbf{r}_v &= 2u \cos(v)\mathbf{j} - u \sin(v)\mathbf{k} \\
\mathbf{r}_u(1, 0) &= 2\mathbf{i} + \mathbf{k} \\
\mathbf{r}_v(1, 0) &= 2\mathbf{j} \\
\mathbf{r}_u \times \mathbf{r}_v &= -2\mathbf{i} + 4\mathbf{k}
\end{align*}
\]
we see that an equation for the tangent plane is
\[-2(x - 1) + 0(y - 0) + 4(z - 1) = 0 \]
or
\[-2x + 4z = 2 \]
or
\[x - 2z = -1.\]

37. For the parametric surface
\[ \mathbf{r}(u, v) = u\mathbf{i} + \ln(uv)\mathbf{j} + v\mathbf{k} \]
the point \((1, 0, 1)\) corresponds to the parameter values \((u, v) = (1, 1)\). Since
\[
\begin{align*}
\mathbf{r}_u &= \mathbf{i} + \frac{1}{u}\mathbf{j} \\
\mathbf{r}_v &= \frac{1}{v}\mathbf{j} + \mathbf{k} \\
\mathbf{r}_u(1, 1) &= \mathbf{i} + \mathbf{j} \\
\mathbf{r}_v(1, 1) &= \mathbf{j} + \mathbf{k} \\
\mathbf{r}_u \times \mathbf{r}_v &= \mathbf{i} - \mathbf{j} + \mathbf{k}
\end{align*}
\]
we see that an equation for the tangent plane is
\[(x - 1) - 1(y - 0) + 1(z - 1) = 0 \]
or
\[x - 1 - y + z - 1 = 0 \]
or
\[x - y + z = 2.\]