1. For

\[ z = \sin(x) \cos(y) \]
\[ x = \pi t \]
\[ y = \sqrt{t} \]

we have

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

\[
= (\cos(x) \cos(y)) (\pi) + (-\sin(x) \sin(y)) \frac{1}{2\sqrt{t}}
\]

\[
= \pi \cos(x) \cos(y) - \frac{\sin(x) \sin(y)}{2\sqrt{t}}.
\]

3. For

\[ w = xe^{y/z} \]
\[ x = t^2 \]
\[ y = 1 - t \]
\[ z = 1 + 2t, \]

we have

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}
\]

\[
= e^{y/z} \left( 2t \right) + \frac{x}{z} e^{y/z} \left( -1 \right) - \frac{xy}{z^2} e^{y/z} \left( 2 \right)
\]

\[
= e^{y/z} \left( 2t - \frac{x}{z} - \frac{2xy}{z^2} \right).
\]

5. For

\[ z = x^2 + xy + y^2 \]
\[ x = s + t \]
\[ y = st \]

we have

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}
\]

\[
= (2x + y) (1) + (x + 2y) (t)
\]
and
\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]
\[
= (2x + y) \frac{1}{2} + (x + 2y) \frac{s}{\sqrt{s^2 + t^2}}.
\]

7. For
\[
z = e^r \cos (\theta)
\]
\[
r = st
\]
\[
\theta = \sqrt{s^2 + t^2}
\]
we have
\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r} \frac{dr}{ds} + \frac{\partial z}{\partial \theta} \frac{d\theta}{ds}
\]
\[
= e^r \cos (\theta) \frac{t}{\sqrt{s^2 + t^2}} - e^r \sin (\theta) \frac{s}{\sqrt{s^2 + t^2}}.
\]

and
\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r} \frac{dr}{dt} + \frac{\partial z}{\partial \theta} \frac{d\theta}{dt}
\]
\[
= e^r \cos (\theta) \frac{s}{\sqrt{s^2 + t^2}} - e^r \sin (\theta) \frac{t}{\sqrt{s^2 + t^2}}.
\]

9. For
\[
z = f(x, y)
\]
\[
x = g(t)
\]
\[
y = h(t)
\]
we have
\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]
\[
= f_x (x, y) g'(t) + f_y (x, y) h'(t)
\]
so
\[
\frac{dz}{dt} \bigg|_{t=3} = f_x (2, 7) g'(3) + f_y (2, 7) h'(3)
\]
\[
= (6)(5) + (-8)(-4)
\]
\[
= 62.
\]
11. For
\[ u = f(x, y) \]
\[ x = x(r, s, t) \]
\[ y = y(r, s, t) \]
we have
\[ \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} \]
and
\[ \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \]
and
\[ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}. \]

13. For
\[ v = f(p, q, r) \]
\[ p = p(x, y, z) \]
\[ q = q(x, y, z) \]
\[ r = r(x, y, z) \]
we have
\[ \frac{\partial v}{\partial x} = \frac{\partial v}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial v}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial v}{\partial r} \frac{\partial r}{\partial x} \]
and
\[ \frac{\partial v}{\partial y} = \frac{\partial v}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial v}{\partial q} \frac{\partial q}{\partial y} + \frac{\partial v}{\partial r} \frac{\partial r}{\partial y} \]
and
\[ \frac{\partial v}{\partial z} = \frac{\partial v}{\partial p} \frac{\partial p}{\partial z} + \frac{\partial v}{\partial q} \frac{\partial q}{\partial z} + \frac{\partial v}{\partial r} \frac{\partial r}{\partial z}. \]

15. For
\[ w = x^2 + y^2 + z^2 \]
\[ x = st \]
\[ y = s \cos(t) \]
\[ z = s \sin(t) \]
we have
\[ \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} \]
\[ = 2xt + 2y \cos(t) + 2z \sin(t) \]
\[ = 2st^2 + 2s \cos^2(t) + 2s \sin^2(t) \]
\[ = 2st^2 + 2s \]
\[ = 2s (t^2 + 1) \]
so
\[ \frac{\partial w}{\partial s} \bigg|_{s=1,t=0} = 2 \left( 1 \right) \left( 0^2 + 1 \right) = 2. \]

Also,
\[ \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} = \]
\[ = 2xs + 2y \left( -s \sin \left( t \right) \right) + 2z \left( s \cos \left( t \right) \right) = 2s^2t - 2s^2 \sin \left( t \right) \cos \left( t \right) + 2s^2 \sin \left( t \right) \cos \left( t \right) = 2s^2t \]

so
\[ \frac{\partial w}{\partial t} \bigg|_{s=1,t=0} = 2 \left( 1 \right)^2 \left( 0 \right) = 0. \]

17. For
\[ z = y^2 \tan \left( x \right) \]
\[ x = t^2uv \]
\[ y = u + tv^2 \]
we have
\[ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial t} = \]
\[ = y^2 \sec^2 \left( x \right) \left( 2tuv \right) + 2y \tan \left( x \right) \left( tv^2 \right) \]

so
\[ \frac{\partial z}{\partial t} \bigg|_{t=2,u=1,v=0} = \left( 1 \right)^2 \sec^2 \left( 0 \right) \left( 0 \right) + 2 \left( 1 \right) \tan \left( 0 \right) \left( 0 \right) = 0. \]

Also,
\[ \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = \]
\[ = y^2 \sec^2 \left( x \right) \left( t^2v \right) + 2y \tan \left( x \right) \left( 1 \right) \]

so
\[ \frac{\partial z}{\partial u} \bigg|_{t=2,u=1,v=0} = \left( 1 \right)^2 \sec^2 \left( 0 \right) \left( 0 \right) + 2 \left( 1 \right) \tan \left( 0 \right) \left( 1 \right) = 0. \]

Also,
\[ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = \]
\[ = y^2 \sec^2 \left( x \right) \left( t^2u \right) + 2y \tan \left( x \right) \left( 2tv \right) \]

so
\[ \frac{\partial z}{\partial v} \bigg|_{t=2,u=1,v=0} = \left( 1 \right)^2 \sec^2 \left( 0 \right) \left( 4 \right) + 2 \left( 1 \right) \tan \left( 0 \right) \left( 0 \right) = 4. \]
19. For
\[
u = \frac{x + y}{y + z}
\]
\[x = p + r + t\]
\[y = p - r + t\]
\[z = p + r - t\]

we have
\[
\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial p} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial p} = \frac{1}{y + z} (1 + \frac{z - x}{(y + z)^2} (1) - \frac{(x + y)}{(y + z)^2} (1)
\]
and
\[
\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r} = \frac{1}{y + z} (1 + \frac{z - x}{(y + z)^2} (-1) - \frac{(x + y)}{(y + z)^2} (1)
\]
and
\[
\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} = \frac{1}{y + z} (1 + \frac{z - x}{(y + z)^2} (1) - \frac{(x + y)}{(y + z)^2} (-1) .
\]

27. The temperature at the point \((x, y)\) on the bug’s path is \(T(x, y)\). The position of the bug at time \(t\) is
\[
x = \sqrt{1 + t}
\]
\[y = 2 + \frac{1}{3} t.
\]
When \(t = 3\), we have \((x, y) = (2, 3)\).
The rate at which the bug’s temperature is changing at any time \(t\) is
\[
\frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt}
\]
\[= T_x(x, y) \left(\frac{1}{2x}\right) + T_y(x, y) \left(\frac{1}{3}\right)
\]
so at time \(t = 3\) we have
\[
\frac{dT}{dt} \bigg|_{t=3} = T_x(2, 3) \left(\frac{1}{2(2)}\right) + T_y(2, 3) \left(\frac{1}{3}\right)
\]
\[= 4 \left(\frac{1}{4}\right) + (3) \left(\frac{1}{3}\right)
\]
\[= 2.
\]
At time $t = 3$, the bug’s temperature is changing at the rate of $2^\circ C$ per second.

31. (a) The volume of the box is $V = LWH$. Thus, the rate of change of volume with respect to time is

$$\frac{dV}{dt} = \frac{\partial V}{\partial L} \frac{dL}{dt} + \frac{\partial V}{\partial W} \frac{dW}{dt} + \frac{\partial V}{\partial H} \frac{dH}{dt}$$

$$= WH \frac{dL}{dt} + LH \frac{dW}{dt} + LW \frac{dH}{dt}$$

At the particular instant being referred to, we have

$$\frac{dV}{dt} = (2 \text{ m})(2 \text{ m})(2 \text{ m/s})$$
$$+ (1 \text{ m})(2 \text{ m})(2 \text{ m/s})$$
$$+ (1 \text{ m})(2 \text{ m})(-3 \text{ m/s})$$
$$= 6 \text{ m}^3/\text{s}$$

so the volume is changing at the rate of $6 \text{ m}^3/\text{s}$.

(b) The surface area of the box is

$$A = 2(LW + LH + WH)$$

Thus, the rate of change of the surface area with respect to time is

$$\frac{dA}{dt} = \frac{\partial A}{\partial L} \frac{dL}{dt} + \frac{\partial A}{\partial W} \frac{dW}{dt} + \frac{\partial A}{\partial H} \frac{dH}{dt}$$

$$= 2(W + H) \frac{dL}{dt} + 2(L + H) \frac{dW}{dt} + 2(L + W) \frac{dH}{dt}$$
$$= 2(2 + 2)(2) + 2(1 + 2)(2) + 2(1 + 2)(-3)$$
$$= 16 + 12 - 18$$
$$= 10.$$ 

Thus the surface area is changing at the rate of $10 \text{ m}^2/\text{s}$.

(c) The length of a diagonal of the box is

$$D = \sqrt{L^2 + W^2 + H^2}$$

Thus, the rate of change of the length of this diagonal with respect to time is

$$\frac{dD}{dt} = \frac{\partial D}{\partial L} \frac{dL}{dt} + \frac{\partial D}{\partial W} \frac{dW}{dt} + \frac{\partial D}{\partial H} \frac{dH}{dt}$$

$$= \frac{L}{D} \frac{dL}{dt} + \frac{W}{D} \frac{dW}{dt} + \frac{H}{D} \frac{dH}{dt}$$

$$= \left( \frac{1}{3} \right)(2) + \left( \frac{2}{3} \right)(2) + \left( \frac{2}{3} \right)(-3)$$
$$= \frac{2}{3} + \frac{4}{3} - 2$$
$$= 0.$$
Thus the length of the diagonal is not changing!

39. For 

\[ z = f(x + at) + g(x - at) , \]

we have

\[ \frac{\partial z}{\partial t} = af'(x + at) - ag'(x - at) \]
\[ \frac{\partial^2 z}{\partial t^2} = a^2 f''(x + at) + a^2 g''(x - at) . \]

Also,

\[ \frac{\partial z}{\partial x} = f'(x + at) + g'(x - at) \]
\[ \frac{\partial^2 z}{\partial x^2} = f''(x + at) + g''(x - at) . \]

We can thus see that \( z \) satisfies the wave equation,

\[ \frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \]