Answers and Solutions to Section 11.6 Homework
Problems 1-17 (odd), 33, 35, 37, 41, and 53.
(A few of these problems are different from the ones with corresponding
numbers in the book.)

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1. It look like the eye of the hurricane is northeast of Raleigh and it looks like
the barometric pressure at Raleigh is about 994 millibars. If we go about
10 miles northeast of Raleigh, the barometric pressure is 992 millibars.
Thus the directional derivative of the pressure function is about
$$\frac{992 - 994}{10} \approx -0.2 \text{ millibars/mile}.$$ (This is, of course, not very precise because I am estimating distances, etc., by looking at a picture.)

3. Skip it.

5. For \( f(x, y) = \sqrt{5x - 4y} \) we have
\[
\nabla f(x, y) = f_x(x, y)i + f_y(x, y)j = \frac{5}{2\sqrt{5x - 4y}}i - \frac{2}{\sqrt{5x - 4y}}j.
\]
so
\[
\nabla f(4, 1) = \frac{5}{8}i - \frac{1}{2}j.
\]
The directional derivative of \( f \) in the direction of the unit vector, \( u = \cos(-\pi/6)i + \sin(-\pi/6)j \) is
\[
D_u f(4, 1) = \nabla f(4, 1) \cdot u = \left( \frac{5}{8} \right) \left( \frac{\sqrt{3}}{2} \right) + \left( -\frac{1}{2} \right) \left( -\frac{1}{2} \right)
= \frac{5\sqrt{3}}{16} + \frac{1}{4}.
\]

7. For \( f(x, y) = 5xy^2 - 4x^3y \), we have
\[
\nabla f(x, y) = (5y^2 - 12x^2y)i + (10xy - 4x^3)j
\]
\[
\nabla f(1, 2) = -4i + 16j.
\]
Also for \( u = \frac{5}{13}i + \frac{12}{13}j \), we have
\[
D_u f(1, 2) = \nabla f(1, 2) \cdot u = -4 \left( \frac{5}{13} \right) + 16 \left( \frac{12}{13} \right)
= \frac{172}{13}.
\]
9. For \( f(x, y, z) = xy^2z^3 \), we have
\[
\nabla f(x, y, z) = (y^2z^3) \mathbf{i} + (2xyz^3) \mathbf{j} + (3xy^2z^2) \mathbf{k}
\]
\[
\nabla f(1, -2, 1) = 4\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}
\]
Also for \( \mathbf{u} = \frac{1}{\sqrt{3}} \mathbf{i} - \frac{2}{\sqrt{3}} \mathbf{j} + \frac{1}{\sqrt{3}} \mathbf{k} \), we have
\[
D_\mathbf{u} f(1, -2, 1) = \nabla f(1, -2, 1) \cdot \mathbf{u}
\]
\[
= 4 \left( \frac{1}{\sqrt{3}} \right) - 4 \left( -\frac{1}{\sqrt{3}} \right) + 12 \left( \frac{1}{\sqrt{3}} \right)
\]
\[
= \frac{20}{\sqrt{3}}.
\]
11. For
\[
f(x, y) = 1 + 2x\sqrt{y},
\]
we have
\[
\nabla f(x, y) = 2\sqrt{y} \mathbf{i} + \frac{x}{\sqrt{y}} \mathbf{j}.
\]
Thus
\[
\nabla f(3, 4) = 4\mathbf{i} + \frac{3}{2} \mathbf{j}.
\]
A unit vector in the direction of \( \mathbf{v} = 4\mathbf{i} - 3\mathbf{j} \) is \( \mathbf{u} = \frac{4}{5} \mathbf{i} - \frac{3}{5} \mathbf{j} \). Therefore
\[
D_\mathbf{u} f(3, 4) = \nabla f(3, 4) \cdot \mathbf{u}
\]
\[
= 4 \left( \frac{4}{5} \right) + \frac{3}{2} \left( -\frac{3}{5} \right)
\]
\[
= \frac{23}{10}.
\]
13. For
\[
f(x, y, z) = \sqrt{x^2 + y^2 + z^2},
\]
we have
\[
\nabla f(x, y, z) = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \mathbf{k}.
\]
Thus
\[
\nabla f(1, 2, -2) = \frac{1}{3} \mathbf{i} + \frac{2}{3} \mathbf{j} - \frac{2}{3} \mathbf{k}.
\]
A unit vector in the direction of \( \mathbf{v} = -6\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} \) is \( \mathbf{u} = -\frac{2}{3} \mathbf{i} + \frac{2}{3} \mathbf{j} - \frac{1}{3} \mathbf{k} \). Therefore
\[
D_\mathbf{u} f(1, 2, -2) = \nabla f(1, 2, -2) \cdot \mathbf{u}
\]
\[
= \frac{1}{3} \left( \frac{2}{3} \right) + \frac{2}{3} \left( \frac{2}{3} \right) - \frac{2}{3} \left( -\frac{1}{3} \right)
\]
\[
= \frac{4}{9}.
\]
15. For 
\[ f(x, y, z) = x \arctan \left( \frac{y}{z} \right), \]
we have 
\[ \nabla f(x, y, z) = \arctan \left( \frac{y}{z} \right) \mathbf{i} + \frac{x}{z} \left(1 + \left(\frac{y}{z}\right)^2\right)^{-1} \frac{xy}{z^2} \mathbf{j} - \frac{x}{z^2} \left(1 + \left(\frac{y}{z}\right)^2\right)^{-1} \mathbf{k}. \]
Thus 
\[ \nabla f(1, 2, -2) = -\frac{\pi}{4} \mathbf{i} - \frac{1}{4} \mathbf{j} - \frac{1}{4} \mathbf{k}. \]
A unit vector in the direction of \( \mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k} \) is \( \mathbf{u} = \frac{1}{\sqrt{3}} \mathbf{i} + \frac{1}{\sqrt{3}} \mathbf{j} - \frac{1}{\sqrt{3}} \mathbf{k} \). Therefore 
\[ D_{\mathbf{u}} f(1, 2, -2) = \nabla f(1, 2, -2) \cdot \mathbf{u} = -\frac{\sqrt{3} \pi}{12}. \]

17. For \( f(x, y) = \sqrt{xy} \) we have 
\[ \nabla f(x, y) = \frac{1}{2} \sqrt{\frac{y}{x}} \mathbf{i} + \frac{1}{2} \sqrt{\frac{x}{y}} \mathbf{j}. \]
Thus 
\[ \nabla f(2, 8) = \mathbf{i} + \frac{1}{4} \mathbf{j}. \]
The vector pointing from \( P(2, 8) \) to \( Q(5, 4) \) is \( \overrightarrow{PQ} = 3 \mathbf{i} - 4 \mathbf{j} \) and a unit vector in this direction is \( \mathbf{u} = \frac{3}{5} \mathbf{i} - \frac{4}{5} \mathbf{j} \). Thus 
\[ D_{\mathbf{u}} f(2, 8) = \nabla f(2, 4) \cdot \mathbf{u} = (1) \left( \frac{3}{5} \right) + \left( \frac{1}{4} \right) \left( -\frac{4}{5} \right) = \frac{2}{5}. \]
\( \nabla (au + bv) = \frac{\partial}{\partial x} (au + bv) \mathbf{i} + \frac{\partial}{\partial y} (au + bv) \mathbf{j} \)

\[
= \left( a \frac{\partial u}{\partial x} + b \frac{\partial v}{\partial x} \right) \mathbf{i} + \left( a \frac{\partial u}{\partial y} + b \frac{\partial v}{\partial y} \right) \mathbf{j} \\
= \left( a \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) \mathbf{i} + \left( b \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right) \mathbf{j} \\
= a \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) + b \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right) \\
= a \nabla u + b \nabla v.
\]

(b)

\( \nabla (uv) = \frac{\partial}{\partial x} (uv) \mathbf{i} + \frac{\partial}{\partial y} (uv) \mathbf{j} \)

\[
= \left( u \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} v \right) \mathbf{i} + \left( u \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} v \right) \mathbf{j} \\
= \left( \frac{\partial u}{\partial x} i + \frac{\partial v}{\partial x} j \right) + \left( \frac{\partial u}{\partial y} i + \frac{\partial v}{\partial y} j \right) \\
= u \left( \frac{\partial v}{\partial x} i + \frac{\partial v}{\partial y} j \right) + v \left( \frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j \right) \\
= u \nabla v + v \nabla u.
\]

(c)

\( \nabla \left( \frac{u}{v} \right) = \frac{\partial}{\partial x} \left( \frac{u}{v} \right) \mathbf{i} + \frac{\partial}{\partial y} \left( \frac{u}{v} \right) \mathbf{j} \)

\[
= \left( \frac{v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}}{v^2} \right) \mathbf{i} + \left( \frac{v \frac{\partial u}{\partial y} - u \frac{\partial v}{\partial y}}{v^2} \right) \mathbf{j} \\
= \frac{1}{v^2} \left( \left( v \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial y} \right) - \left( u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} \right) \right) \\
= \frac{1}{v^2} \left( v \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) - u \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \\
= \frac{v \nabla u - u \nabla v}{v^2}.
\]
(d) 
\[
\nabla (u^n) = \frac{\partial}{\partial x} (u^n) \mathbf{i} + \frac{\partial}{\partial y} (u^n) \mathbf{j} \\
= nu^{n-1} \frac{\partial u}{\partial x} \mathbf{i} + nu^{n-1} \frac{\partial u}{\partial y} \mathbf{j} \\
= nu^{n-1} \left( \frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} \right) \\
= n u^{n-1} \nabla u.
\]

35. Since the surface \( x^2 + 2y^2 + 3z^2 = 21 \) is a level surface of the function \( f(x, y, z) = x^2 + 2y^2 + 3z^2 \), we know that the gradient vector, \( \nabla f(4, -1, 1) \) is normal (perpendicular) to the surface at the point \((4, -1, 1)\). This gradient vector will serve both as a normal vector for the tangent plane and as a direction vector for the normal line.

\[
\nabla f(x, y, z) = 2xi + 4yj + 6zk
\]

and

\[
\nabla f(4, -1, 1) = 8i - 4j + 6k
\]

so the tangent plane has equation

\[
8 (x - 4) - 4 (y + 1) + 6 (z - 1) = 0
\]

or more simply

\[
4 (x - 4) - 2 (y + 1) + 3 (z - 1) = 0
\]

or

\[
4x - 2y + 3z = 21
\]

and the normal line has symmetric equations

\[
\frac{x - 4}{4} = \frac{y + 1}{-2} = \frac{z - 1}{3}.
\]

37. Since the surface \( z = 1 - xe^y \cos(z) = 0 \) is a level surface of the function \( f(x, y, z) = z + 1 - xe^y \cos(z) \),

we know that the gradient vector, \( \nabla f(1, 0, 0) \) is normal (perpendicular) to the surface at the point \((1, 0, 0)\). This gradient vector will serve both as a normal vector for the tangent plane and as a direction vector for the normal line.

\[
\nabla f(x, y, z) = -e^y \cos(z) \mathbf{i} - xe^y \cos(z) \mathbf{j} + (1 + xe^y \sin(z)) \mathbf{k}
\]

and

\[
\nabla f(1, 0, 0) = -\mathbf{i} - \mathbf{j} + \mathbf{k}
\]
so the tangent plane has equation
\[-1(x - 1) - 1(y - 0) + 1(z - 0) = 0\]
or more simply
\[x + y - z = 1\]
and the normal line has symmetric equations
\[x - 1 = y = -z.\]

41. For \( f(x, y) = x^2 + 4y^2 \), we have
\[\nabla f(x, y) = 2xi + 8yj\]
and
\[\nabla f(2, 1) = 4i + 8j.\]
This gradient vector is perpendicular to the level curve \( f(x, y) = 8 \). Thus, the tangent line to the level curve has equation
\[\nabla f(2, 1) \cdot ((x - 2)i + (y - 1)j) = 0\]
or
\[(4i + 8j) \cdot ((x - 2)i + (y - 1)j) = 0\]
or
\[4(x - 2) + 8(y - 1) = 0\]
\[y = \frac{-1}{2}x + 2.\]
The curve \( x^2 + 4y^2 = 8 \), the gradient vector \( \nabla f(2, 1) = 4i + 8j \), and the tangent line \( y = -\frac{1}{2}x + 2 \) are pictured below.
The problem here is as follows: If \( u \) and \( v \) are non–parallel unit vectors and we know \( D_u f \) and \( D_v f \), can we express \( \nabla f \) in terms of \( D_u f \) and \( D_v f \)? I actually spent a good bit of time working on this problem and came up with a solution that contains some interesting results along the way. This is not the shortest solution, of course, but the intermediate results are interesting in themselves. We will get to the solution after proving two lemmas.

**Lemma 1:** If \( u \) and \( v \) are any two non–parallel unit vectors and \( w \) is any vector, then

\[
    w = u \cdot w - (u \cdot v) (v \cdot w) \quad \frac{1}{1 - (u \cdot v)^2} u + v \cdot w - (u \cdot v) (u \cdot w) \quad \frac{1}{1 - (u \cdot v)^2} v.
\]

**Proof of Lemma 1** First, note that since \( u \) and \( v \) are non–parallel unit vectors, then \( u \cdot v \neq 1 \) and \( u \cdot v \neq -1 \). Thus \( (u \cdot v)^2 \neq 1 \) and the denominators in the formula we are trying to establish is not 0.

The problem here is to find scalars \( x \) and \( y \) such that \( w = xu + yv \). First we observe that if \( w = xu + yv \), then

\[
    u \cdot w = xu \cdot u + yu \cdot v = x + u \cdot vy
\]
and likewise 
\[ \mathbf{v} \cdot \mathbf{w} = x \mathbf{u} \cdot \mathbf{v} + y \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} x + y. \]

We must now solve the linear system
\[ x + \mathbf{u} \cdot \mathbf{v} y = \mathbf{u} \cdot \mathbf{w} \]
\[ \mathbf{u} \cdot \mathbf{v} x + y = \mathbf{v} \cdot \mathbf{w} \]
for \( x \) and \( y \). If we replace the second equation with \(-\mathbf{u} \cdot \mathbf{v}\) times the first equation, we obtain the system
\[ x + \mathbf{u} \cdot \mathbf{v} y = \mathbf{u} \cdot \mathbf{w} \]
\[ \left(1 - (\mathbf{u} \cdot \mathbf{v})^2\right) y = \mathbf{v} \cdot \mathbf{w} - (\mathbf{u} \cdot \mathbf{v}) (\mathbf{u} \cdot \mathbf{w}) \]
from which we see that
\[ y = \frac{\mathbf{v} \cdot \mathbf{w} - (\mathbf{u} \cdot \mathbf{v}) (\mathbf{u} \cdot \mathbf{w})}{1 - (\mathbf{u} \cdot \mathbf{v})^2}. \]

By similar reasoning, we see that
\[ x = \frac{\mathbf{u} \cdot \mathbf{w} - (\mathbf{u} \cdot \mathbf{v}) (\mathbf{v} \cdot \mathbf{w})}{1 - (\mathbf{u} \cdot \mathbf{v})^2} \]
and this proves the lemma.

**Lemma 2:** If \( \mathbf{u} \) and \( \mathbf{v} \) are any two non–parallel unit vectors and \( \mathbf{a} \) and \( \mathbf{b} \) are vectors such that
\[ \mathbf{a} \cdot \mathbf{u} = \mathbf{b} \cdot \mathbf{u} \]
and
\[ \mathbf{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}, \]
then \( \mathbf{a} = \mathbf{b} \).

**Proof of Lemma 2:** Using Lemma 1, we obtain
\[
\mathbf{a} = \frac{(\mathbf{u} \cdot \mathbf{a} - (\mathbf{u} \cdot \mathbf{v}) (\mathbf{v} \cdot \mathbf{a}) \mathbf{u} + (\mathbf{v} \cdot \mathbf{a} - (\mathbf{u} \cdot \mathbf{v}) (\mathbf{u} \cdot \mathbf{a}) \mathbf{v}}{1 - (\mathbf{u} \cdot \mathbf{v})^2}
\]
\[ = \frac{\mathbf{u} \cdot \mathbf{b} - (\mathbf{u} \cdot \mathbf{v}) (\mathbf{v} \cdot \mathbf{b}) \mathbf{u} + (\mathbf{v} \cdot \mathbf{b} - (\mathbf{u} \cdot \mathbf{v}) (\mathbf{u} \cdot \mathbf{b}) \mathbf{v}}{1 - (\mathbf{u} \cdot \mathbf{v})^2}
\]
\[ = \mathbf{b}. \]

We now get to the main problem: We are given two non–parallel unit vectors \( \mathbf{u} \) and \( \mathbf{v} \) and we suppose that we know what \( D_u f \) and \( D_v f \) are. We would like to find \( \nabla f \) in terms of this give information.

By Lemma 1, we see that
\[
\nabla f = \frac{\mathbf{u} \cdot \nabla f - (\mathbf{u} \cdot \mathbf{v}) (\mathbf{v} \cdot \nabla f)}{1 - (\mathbf{u} \cdot \mathbf{v})^2} \mathbf{u} + \frac{\mathbf{v} \cdot \nabla f - (\mathbf{u} \cdot \mathbf{v}) (\mathbf{u} \cdot \nabla f)}{1 - (\mathbf{u} \cdot \mathbf{v})^2} \mathbf{v}
\]
\[ = \frac{D_u f - (\mathbf{u} \cdot \mathbf{v}) D_v f}{1 - (\mathbf{u} \cdot \mathbf{v})^2} \mathbf{u} + \frac{D_v f - (\mathbf{u} \cdot \mathbf{v}) D_u f}{1 - (\mathbf{u} \cdot \mathbf{v})^2} \mathbf{v}. \]
This solves our problem and we can use Lemma 2 to verify that our solution is correct: Note that \( \mathbf{u} \) and \( \mathbf{v} \) are non–parallel unit vectors and define

\[
\mathbf{a} = \nabla f
\]
and

\[
\mathbf{b} = \frac{D_u f - (\mathbf{u} \cdot \mathbf{v}) D_v f}{1 - (\mathbf{u} \cdot \mathbf{v})^2} \mathbf{u} + \frac{D_v f - (\mathbf{u} \cdot \mathbf{v}) D_u f}{1 - (\mathbf{u} \cdot \mathbf{v})^2} \mathbf{v}.
\]

Then

\[
\mathbf{a} \cdot \mathbf{u} = \nabla f \cdot \mathbf{u} = D_u f
\]
and

\[
\mathbf{b} \cdot \mathbf{u} = \frac{D_u f - (\mathbf{u} \cdot \mathbf{v}) D_v f}{1 - (\mathbf{u} \cdot \mathbf{v})^2} + \frac{(\mathbf{u} \cdot \mathbf{v}) D_v f - (\mathbf{u} \cdot \mathbf{v})^2 D_u f}{1 - (\mathbf{u} \cdot \mathbf{v})^2}
= D_u f
\]
which shows that \( \mathbf{a} \cdot \mathbf{u} = \mathbf{b} \cdot \mathbf{u} \).

Also

\[
\mathbf{a} \cdot \mathbf{v} = \nabla f \cdot \mathbf{v} = D_v f
\]
and

\[
\mathbf{b} \cdot \mathbf{v} = \frac{(\mathbf{u} \cdot \mathbf{v}) D_u f - (\mathbf{u} \cdot \mathbf{v})^2 D_v f}{1 - (\mathbf{u} \cdot \mathbf{v})^2} + \frac{D_v f - (\mathbf{u} \cdot \mathbf{v}) D_u f}{1 - (\mathbf{u} \cdot \mathbf{v})^2}
= D_v f
\]
which shows that \( \mathbf{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v} \).

By Lemma 2, we conclude that \( \mathbf{a} = \mathbf{b} \).

Let us verify this result by checking a concrete example: For the function \( f(x, y) = x^2 + 4y^3 \), we have

\[
\nabla f(x, y) = 2x\mathbf{i} + 12y^2\mathbf{j}.
\]

Thus

\[
\nabla f(3, 1) = 6\mathbf{i} + 12\mathbf{j}.
\]

Now, let us take the unit vectors

\[
\mathbf{u} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}
\]
and

\[
\mathbf{v} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.
\]

For these unit vectors, we have

\[
D_{\mathbf{u}} f(3, 1) = \nabla f \cdot \mathbf{u}
= 6\left(\frac{\sqrt{2}}{2}\right) + 12\left(-\frac{\sqrt{2}}{2}\right)
= -3\sqrt{2}
\]

9
and

\[ D_v f(3, 1) = \nabla f \cdot v \]
\[ = 6 \left( \frac{3}{5} \right) + 12 \left( \frac{4}{5} \right) \]
\[ = \frac{66}{5}. \]

Note that

\[ u \cdot v = \left( \frac{\sqrt{2}}{2} \right) \left( \frac{3}{5} \right) + \left( -\frac{\sqrt{2}}{2} \right) \left( \frac{4}{5} \right) \]
\[ = -\frac{\sqrt{2}}{10} \]

and thus

\[ \frac{D_{u,v} f - (u \cdot v) D_v f}{1 - (u \cdot v)^2} u + \frac{D_{u,v} f - (u \cdot v) D_u f}{1 - (u \cdot v)^2} v \]
\[ = -\frac{3\sqrt{2}}{7} - \left( -\frac{\sqrt{2}}{10} \right) \left( \frac{66}{5} \right) \left( \frac{\sqrt{2}}{2} i - \frac{\sqrt{2}}{2} j \right) \]
\[ + \frac{66}{5} \left( -\frac{\sqrt{2}}{10} \right) \left( -3\sqrt{2} \right) \left( \frac{3}{5} i + \frac{4}{5} j \right) \]
\[ = -\frac{12\sqrt{2}}{7} \left( \frac{\sqrt{2}}{2} i - \frac{\sqrt{2}}{2} j \right) + \frac{90}{7} \left( \frac{3}{5} i + \frac{4}{5} j \right) \]
\[ = \left( -\frac{12}{7} + \frac{54}{7} \right) i + \left( \frac{12}{7} + \frac{72}{7} \right) j \]
\[ = 6i + 12j \]
\[ = \nabla f(3, 1). \]