1. By looking at the picture in the book, we see that
\[ \int_C \nabla f \cdot dr = 50 - 10 = 40. \]

3. For the vector field
\[ \mathbf{F}(x,y) = (6x + 5y)i + (5x + 4y)j, \]
we have
\[ \frac{\partial P}{\partial y} = 5 \]
\[ \frac{\partial Q}{\partial x} = 5. \]
Since the (implied) domain of \( \mathbf{F} \) is \( \mathbb{R}^2 \), which is a simply–connected set, we conclude that \( \mathbf{F} \) is conservative.

To find \( f \) such that \( \nabla f = \mathbf{F} \), we begin with
\[ f_x(x,y) = 6x + 5y. \]
This gives us
\[ f(x,y) = 3x^2 + 5xy + h(y). \]
Differentiation with respect to \( y \) then gives us
\[ f_y(x,y) = 5x + h'(y). \]
However, we must also have
\[ f_y(x,y) = 5x + 4y. \]
Thus
\[ h'(y) = 4y \]
which means that
\[ h(y) = 2y^2 + C. \]
Since we are only looking for a single potential function, we might as well take \( C = 0 \). We thus obtain
\[ f(x,y) = 3x^2 + 5xy + 2y^2. \]
Let us check that this is correct:
\[ \nabla f(x,y) = (6x + 5y)i + (5x + 4y)j = \mathbf{F}(x,y). \]

5. For the vector field
\[ \mathbf{F}(x,y) = xe^y i + ye^x j, \]
we have
Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, we conclude that the vector field $\mathbf{F}$ is not conservative.

7. For the vector field
   \[ \mathbf{F}(x, y) = (2x \cos(y) - y \cos(x))\mathbf{i} + (-x^2 \sin(y) - \sin(x))\mathbf{j}, \]
   we have
   \[ \frac{\partial P}{\partial y} = -2x \sin(y) - \cos(x) \]
   \[ \frac{\partial Q}{\partial x} = -2x \sin(y) - \cos(x). \]
   Since the (implied) domain of $\mathbf{F}$ is $\mathbb{R}^2$, which is a simply–connected set, we conclude that $\mathbf{F}$ is conservative.
   To find $f$ such that $\nabla f = \mathbf{F}$, we begin with
   \[ f_x(x, y) = 2x \cos(y) - y \cos(x). \]
   This gives us
   \[ f(x, y) = x^2 \cos(y) - y \sin(x) + h(y). \]
   Differentiation with respect to $y$ then gives us
   \[ f_y(x, y) = -x^2 \sin(y) - \sin(x) + h'(y). \]
   However, we must also have
   \[ f_y(x, y) = -x^2 \sin(y) - \sin(x). \]
   Thus
   \[ h'(y) = 0 \]
   which means that
   \[ h(y) = C. \]
   Since we are only looking for a single potential function, we might as well take $C = 0$. We thus obtain
   \[ f(x, y) = x^2 \cos(y) - y \sin(x). \]
   Let us check that this is correct:
   \[ \nabla f(x, y) = (2x \cos(y) - y \cos(x))\mathbf{i} + (-x^2 \sin(y) - \sin(x))\mathbf{j} \]
   \[ = \mathbf{F}(x, y). \]

9. For the vector field
   \[ \mathbf{F}(x, y) = (ye^x + \sin(y))\mathbf{i} + (e^x + x \cos(y))\mathbf{j}, \]
   we have
   \[ \frac{\partial P}{\partial y} = e^x + \cos(y) \]
   \[ \frac{\partial Q}{\partial x} = e^x + \cos(y). \]
Since the (implied) domain of $F$ is $\mathbb{R}^2$, which is a simply–connected set, we conclude that $F$ is conservative.

To find $f$ such that $\nabla f = F$, we begin with

$$f_x(x,y) = ye^x + \sin(y).$$

This gives us

$$f(x,y) = ye^x + x\sin(y) + h(y).$$

Differentiation with respect to $y$ then gives us

$$f_y(x,y) = e^x + x\cos(y) + h'(y).$$

However, we must also have

$$f_y(x,y) = e^x + x\cos(y).$$

Thus

$$h'(y) = 0$$

which means that

$$h(y) = C.$$

Since we are only looking for a single potential function, we might as well take $C = 0$. We thus obtain

$$f(x,y) = ye^x + x\sin(y).$$

Let us check that this is correct:

$$\nabla f(x,y) = (ye^x + \sin(y))\mathbf{i} + (e^x + x\cos(y))\mathbf{j} = F(x,y).$$

11. The vector field $F(x,y) = 2xy\mathbf{i} + x^2\mathbf{j}$ is conservative, so all line integrals of $F$ are path independent. A potential function for $F$ is $f(x,y) = x^2y$ so, by the Fundamental Theorem for line integrals, the integral of $F$ over any path beginning at $(1,2)$ and ending at $(3,2)$ is

$$\int F \cdot dr = \int \nabla f \cdot dr$$

$$= f(3,2) - f(1,2)$$

$$= 3^2(2) - 1^2(2)$$

$$= 16.$$  

13. First we find a potential function, $f$, for the vector field

$$F(x,y) = x^3y^4\mathbf{i} + x^4y^3\mathbf{j}.$$  

The potential function must satisfy

$$f_x(x,y) = x^3y^4$$

and thus

$$f(x,y) = \frac{1}{4}x^4y^4 + h(y)$$

and

$$f_y(x,y) = x^4y^3 + h'(y).$$

Since $f$ must also satisfy
\[ f_y(x,y) = x^4 y^3, \]

we obtain \( h'(y) = 0 \) and thus \( h(y) = C \). Therefore, a potential function for \( \mathbf{F} \) is

\[ f(x,y) = \frac{1}{4} x^4 y^4. \]

The path

\[ \mathbf{r}(t) = \sqrt{t} \mathbf{i} + (1 + t^3) \mathbf{j} \]

\[ 0 \leq t \leq 1 \]

has initial point \( \mathbf{r}(0) = (0,1) \) and terminal point \( \mathbf{r}(1) = (1,2) \). Thus, by the Fundamental Theorem for Line integrals, we have

\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} = f(1,2) - f(0,1) \]

\[ = \frac{1}{4} (1)^4 (2)^4 - \frac{1}{4} (0)^4 (1)^4 \]

\[ = 4. \]

15. First we find a potential function, \( f \), for the vector field

\[ \mathbf{F}(x,y,z) = yz \mathbf{i} + xz \mathbf{j} + (xy + 2z) \mathbf{k}. \]

The potential function must satisfy

\[ f_x(x,y,z) = yz \]

and thus

\[ f(x,y,z) = xyz + h(y,z) \]

and

\[ f_y(x,y,z) = xz + \frac{\partial h}{\partial y}. \]

Since \( f \) must also satisfy

\[ f_y(x,y,z) = xz \]

we see that \( \frac{\partial h}{\partial y} = 0 \) and hence that

\[ h(y,z) = g(z). \]

We now have

\[ f(x,y,z) = xyz + g(z) \]

This tells us that

\[ f_z(x,y,z) = xy + g'(z) \]

Since \( f \) must also satisfy

\[ f_z(x,y,z) = xy + 2z, \]

we obtain \( g'(z) = 2z \) and thus \( g(z) = z^2 + C \). Therefore, a potential function for \( \mathbf{F} \) is

\[ f(x,y,z) = xyz + z^2. \]

By the Fundamental Theorem for Line integrals, we have
First we find a potential function, $f$, for the vector field
\[ \mathbf{F}(x,y,z) = y^2 \cos(z) \mathbf{i} + 2xy \cos(z) \mathbf{j} - xy^2 \sin(z) \mathbf{k}. \]
The potential function must satisfy
\[ f_x(x,y,z) = y^2 \cos(z) \]
and thus
\[ f(x,y,z) = xy^2 \cos(z) + h(y,z) \]
and
\[ f_y(x,y,z) = 2xy \cos(z) + \frac{\partial h}{\partial y}. \]
Since $f$ must also satisfy
\[ f_y(x,y,z) = 2xy \cos(z) \]
we see that $\frac{\partial h}{\partial y} = 0$ and hence that
\[ h(y,z) = g(z). \]
We now have
\[ f(x,y,z) = xy^2 \cos(z) + g(z) \]
This tells us that
\[ f_z(x,y,z) = -xy^2 \sin(z) + g'(z) \]
Since $f$ must also satisfy
\[ f_z(x,y,z) = -xy^2 \sin(z), \]
we obtain $g'(z) = 0$ and thus $g(z) = C$.
Therefore, a potential function for $\mathbf{F}$ is
\[ f(x,y,z) = xy^2 \cos(z). \]
The path
\[ \mathbf{r}(t) = t^2 \mathbf{i} + \sin(t) \mathbf{j} + t \mathbf{k} \]
\[ 0 \leq t \leq \pi \]
has initial point $\mathbf{r}(0) = (0,0,0)$ and terminal point $\mathbf{r}(\pi) = (\pi^2,0,\pi)$.
By the Fundamental Theorem for Line integrals, we have
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = f(\pi^2,0,\pi) - f(0,0,0) \]
\[ = (\pi^2)(0)^2 \cos(\pi) - (0)(0)^2 \cos(0) \]
\[ = 0. \]
19. For the vector field
\[ \mathbf{F}(x,y) = 2x \sin(y) \mathbf{i} + (x^2 \cos(y) - 3y^2) \mathbf{j}, \]
we have
\[ \frac{\partial P}{\partial y} = 2x \cos(y) \]
\[ \frac{\partial Q}{\partial x} = 2x \cos(y). \]
Since the (implied) domain of \( F \) is \( \mathbb{R}^2 \), which is a simply-connected set, we conclude that \( F \) is conservative (and hence that all line integrals of \( F \) are path-independent).
To find \( f \) such that \( \nabla f = F \), we begin with
\[ f_x(x, y) = 2x \sin(y). \]
This gives us
\[ f(x, y) = x^2 \sin(y) + h(y). \]
Differentiation with respect to \( y \) then gives us
\[ f_y(x, y) = x^2 \cos(y) + h'(y). \]
However, we must also have
\[ f_y(x, y) = x^2 \cos(y) - 3y^2. \]
Thus
\[ h'(y) = -3y^2 \]
which means that
\[ h(y) = -y^3 + C. \]
Since we are only looking for a single potential function, we might as well take \( C = 0 \). We thus obtain
\[ f(x, y) = x^2 \sin(y) - y^3. \]
Now, by the Fundamental Theorem for Line Integrals, for a path with initial point \((-1, 0)\) and terminal point \((5, 1)\), we have
\[ \int_C 2x \sin(y) \, dx + (x^2 \cos(y) - 3y^2) \, dy = \int_C F \cdot dr \\
= f(5, 1) - f(-1, 0) \\
= 25 \sin(1) - 1. \]
21. The force field
\[ F(x, y) = x^2 y^3 \mathbf{i} + x^3 y^2 \mathbf{j} \]
is conservative and has potential function
\[ f(x, y) = \frac{1}{3} x^3 y^3. \]
Thus, the work done by this force field in moving an object from the point \((0, 0)\) to the point \((2, 1)\) is
23. The vector field, $\mathbf{F}$, shown in the picture (page 944) is not conservative. If it were, then it would have to satisfy

$$\frac{\partial P}{\partial y}(x, y) = \frac{\partial Q}{\partial x}(x, y)$$

at all points $(x, y)$. However, the picture shows that this is not true.

For example, look at the vertical column of vectors that is the second to the left from the $y$ axis. (Hence $x = x_0$ is constant for all vectors in this column.) If we move from bottom to top (in the direction of increasing $y$) along this column, we see that the vectors make a transition from pointing to the left to pointing to the right. There is some point $(x_0, y_0)$ in the third quadrant at which $\mathbf{F}(x_0, y_0) = \mathbf{0}$. Also, since $P(x_0, y)$ transitions from negative to positive as $y$ increases, we see that

$$\frac{\partial P}{\partial y}(x_0, y_0) > 0.$$

However, if we now look at the horizontal row of vectors with $y = y_0$, we see that these vectors transition from pointing up to pointing down as $x$ increases. This means that

$$\frac{\partial Q}{\partial x}(x_0, y_0) < 0.$$

Therefore it is not true that

$$\frac{\partial P}{\partial y}(x_0, y_0) = \frac{\partial Q}{\partial x}(x_0, y_0).$$

29. The set $\{(x, y) \mid x > 0$ and $y > 0\}$ is open, connected, and simply–connected.

30. The set $\{(x, y) \mid x \neq 0\}$ is open, but not connected or simply–connected.

31. The set $\{(x, y) \mid 1 < x^2 + y^2 < 4\}$ is open and connected, but not simply–connected.

32. The set $\{(x, y) \mid x^2 + y^2 \leq 1$ or $4 \leq x^2 + y^2 \leq 9\}$ is neither open, nor connected, nor simply–connected.

33. Consider the vector field

$$\mathbf{F}(x, y) = \frac{-y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j}$$

with domain $D = \{(x, y) \mid (x, y) \neq (0, 0)\}$. The vector field $\mathbf{F}$ satisfies

$$\int \mathbf{F} \cdot d\mathbf{r} = f(2, 1) - f(0, 0)$$

$$= \frac{1}{3}(2)^3(1)^3 - \frac{1}{3}(0)^3(0)^3$$

$$= \frac{8}{3}.$$
$$\frac{\partial P}{\partial y} = \frac{(x^2 + y^2)(-1) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2 + y^2)(1) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

and thus

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$ 

However, since the domain $D$ is not a simply–connected set, we are **not** guaranteed that the vector field $F$ is conservative. In fact, $F$ is not conservative as we will show by computing line integrals of $F$ over two different paths joining that begin at the point $(1, 0)$ and end at the point $(-1, 0)$.

First, we use the path

$$r(t) = \cos(t)i + \sin(t)j$$

$$0 \leq t \leq \pi.$$ 

(This path traces out the top half of the unit circle counterclockwise.)

For this path, we have

$$\int_C F \cdot dr = \int_0^{\pi} F(r(t)) \cdot r'(t) \, dt$$

$$= \int_0^{\pi} (-\sin(t)i + \cos(t)j) \cdot (-\sin(t)i + \cos(t)j) \, dt$$

$$= \int_0^{\pi} 1 \, dt$$

$$= \pi.$$ 

Next, we use the path

$$r(t) = \cos(t)i - \sin(t)j$$

$$0 \leq t \leq \pi.$$ 

(This path traces out the bottom half of the unit circle clockwise.)

For this path, we have

$$\int_C F \cdot dr = \int_0^{\pi} F(r(t)) \cdot r'(t) \, dt$$

$$= \int_0^{\pi} (\sin(t)i + \cos(t)j) \cdot (-\sin(t)i - \cos(t)j) \, dt$$

$$= \int_0^{\pi} -1 \, dt$$

$$= -\pi.$$