Answers and Solutions to Section 9.5 Homework Problems
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1. (a) True
(b) False
(c) True
(d) False
(e) False
(f) True
(g) False
(h) True
(i) True
(j) False
(k) True

3. vector equation: \( \langle x, y, z \rangle = \langle -2, 4, 10 \rangle + t \langle 3, 1, -8 \rangle \)
   parametric equations:
   \[
   x = -2 + 3t \\
   y = 4 + t \\
   z = 10 - 8t
   \]

5. vector equation: \( \langle x, y, z \rangle = \langle 1, 0, 6 \rangle + t \langle 1, 3, 1 \rangle \)
   parametric equations:
   \[
   x = 1 + t \\
   y = 3t \\
   z = 6 + t
   \]

7. parametric equations:
   \[
   x = 2t \\
   y = \frac{1}{2} + \frac{1}{2} t \\
   z = 1 - 4t.
   \]
   symmetric equations:
   \[
   \frac{x}{2} = \frac{y - \frac{1}{2}}{\frac{1}{2}} = \frac{z - 1}{-4}.
   \]
9. parametric equations:

\[ \begin{align*}
x &= 1 + t \\
y &= -1 + 2t \\
z &= 1 + t
\end{align*} \]

symmetric equations:

\[ x - 1 = \frac{y + 1}{2} = z - 1 \]

11. A direction vector of the line through the points \((-4, -6, 1)\) and \((-2, 0, -3)\) is \(\mathbf{v}_1 = (2, 6, -4)\). A direction vector of the line through the points \((10, 18, 4)\) and \((5, 3, 14)\) is \(\mathbf{v}_2 = (-5, -15, 10)\). Since \(\mathbf{v}_2 = -\frac{5}{2} \mathbf{v}_1\), the lines are parallel.

13. (a)

\[ \frac{x}{2} = \frac{y - 2}{3} = \frac{z + 1}{-7} \]

(b) This line intersects the \(yz\) plane at the point \((0, 2, -1)\).

This line intersects the \(xz\) plane at the point \((-\frac{2}{7}, 0, \frac{11}{7})\).

This line intersects the \(xy\) plane at the point \((-\frac{2}{7}, \frac{11}{7}, 0)\).

15. The line segment from \((2, -1, 4)\) to \((4, 6, 1)\) has vector equation

\[ \langle x, y, z \rangle = (2, -1, 4) + t (2, 7, -3) \]

\[ 0 \leq t \leq 1. \]

17. The lines are parallel.

19. The lines are not parallel because their direction vectors are not parallel.

If the lines intersect, then there is some point \((x, y, z)\) that is on both lines.

This means \((x, y)\) must satisfy both of the equations

\[ \begin{align*}
\frac{x}{1} &= \frac{y - 1}{2} \\
\frac{x - 3}{-4} &= \frac{y - 2}{-3}
\end{align*} \]

or (after some algebra)

\[ \begin{align*}
2x - y &= -1 \\
-3x + 4y &= -1
\end{align*} \]

which implies that \(x = y = -1\). Using the equation for \(L_1\), we obtain

\[ \frac{z - 2}{3} = -1 \]

2
which gives $z = -1$, and using the equation for $L_2$ we obtain
\[ \frac{z - 1}{2} = -1 \]
which also gives $z = -1$. Thus we observe that the lines do intersect and that the point $(-1, -1, -1)$ is the point of intersection.

21. $-2(x - 6) + (y - 3) + 5(z - 2) = 0$

23. $2x - y + 3z = 0$

25. The plane through the points $(0, 1, 1), (1, 0, 1)$, and $(1, 1, 0)$ is parallel to the vectors $\mathbf{v}_1 = \mathbf{i} - \mathbf{j}$ and $\mathbf{v}_2 = \mathbf{i} - \mathbf{k}$. Thus, a vector normal to this plane is
\[ \mathbf{v}_1 \times \mathbf{v}_2 = (\mathbf{i} - \mathbf{j}) \times (\mathbf{i} - \mathbf{k}) \]
\[ = \mathbf{i} \times \mathbf{i} - \mathbf{i} \times \mathbf{k} - \mathbf{j} \times \mathbf{i} + \mathbf{j} \times \mathbf{k} \]
\[ = 0 + \mathbf{j} + \mathbf{k} + \mathbf{i} \]
\[ = \mathbf{i} + \mathbf{j} + \mathbf{k}. \]

An equation for the plane is thus
\[ (x - 0) + (y - 1) + (z - 1) = 0 \]
or
\[ x + y + z = 2. \]

(It is easily checked that this plane contains the three given points.)

27. Since the plane in question contains the points $A(6, 0, -2), B(4, 3, 7)$, and $C(6, -2, 3)$ (the latter two points being obtained by setting $t = 0$ and $t = -1$ in the parametric equations of the given line), we can obtain a normal vector for the plane as follows:
\[ \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} \]
\[ = (-2\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}) \times (-2\mathbf{j} + 5\mathbf{k}) \]
\[ = 4\mathbf{k} + 10\mathbf{j} + 15\mathbf{i} + 18\mathbf{i} \]
\[ = 33\mathbf{i} + 10\mathbf{j} + 4\mathbf{k}. \]

An equation of the plane is thus
\[ 33(x - 6) + 10y + 4(z + 2) = 0 \]
or
\[ 33x + 10y + 4z = 190. \]
29. Any point, \((x, y, z)\), that lies in both planes must satisfy both of the equations

\[
\begin{align*}
x + y - z &= 2 \\
2x - y + 3z &= 1
\end{align*}
\]

and hence must satisfy both

\[
\begin{align*}
x + y - z &= 2 \\
-3y + 5z &= -3
\end{align*}
\]

and hence must satisfy both

\[
\begin{align*}
x + \frac{2}{3}z &= 1 \\
-3y + 5z &= -3.
\end{align*}
\]

From this, it can be see that two points lying on the line of intersection of the planes are \((1, 1, 0)\) and \((-1, 6, 3)\). We are looking for the plane that is parallel to the vectors \(\mathbf{v}_1 = \langle -1 - 1, 6 - 1, 3 - 0 \rangle = \langle -2, 5, 3 \rangle\) and \(\mathbf{v}_2 = \langle -1 - 1, 2 - 1, 1 - 0 \rangle = \langle -2, 1, 1 \rangle\). A normal vector for this plane is

\[
\mathbf{v}_1 \times \mathbf{v}_2 = (-2i + 5j + 3k) \times (-2i + j + k) \\
= -2i \times j - 2i \times k - 10j \times i + 5j \times k - 6k \times i + 3k \times j \\
= -2k + 2j + 10k + 5i - 6j - 3i \\
= 2i - 4j + 8k.
\]

An equation for this plane is

\[
2(x + 1) - 4(y - 2) + 8(z - 1) = 0.
\]

31. Solving

\[
2(1 + 2t) + (-1) - (t) + 5 = 0,
\]

we obtain \(t = -2\). The line intersects the plane at the point \((-3, -1, -2)\).

33. The plane with equation \(x + y + z = 1\) has normal vector \(\mathbf{n}_1 = i + j + k\). The plane with equation \(x - y + z = 1\) has normal vector \(\mathbf{n}_2 = i - j + k\). Since

\[
\mathbf{n}_1 \cdot \mathbf{n}_2 = (i + j + k) \cdot (i - j + k) \\
= 1 - 1 + 1 \\
\neq 0
\]

the planes are not perpendicular.
Since

\[ \mathbf{n}_1 \times \mathbf{n}_2 = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k}) \]
\[ = 0 - \mathbf{k} - \mathbf{j} + \mathbf{0} + \mathbf{i} - \mathbf{j} + \mathbf{i} + \mathbf{0} \]
\[ = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \]
\[ \neq \mathbf{0}, \]

the planes are not parallel.

If \( \theta \) is the angle between the two planes, then

\[ \cos (\theta) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \frac{1}{3}. \]

The angle between the planes is thus

\[ \theta = \arccos \left( \frac{1}{3} \right) \approx 70.53^\circ. \]

37. A normal vector for the plane \( x + y - z = 2 \) is \( \mathbf{n}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k} \). A normal vector for the plane \( 3x - 4y + 5z = 6 \) is \( \mathbf{n}_2 = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k} \). Since the line of intersection of these two planes must be orthogonal to both \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \), we can find a direction vector, \( \mathbf{v} \), for this line as follows:

\[ \mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 \]
\[ = (\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) \]
\[ = -4\mathbf{i} \times \mathbf{j} + 5\mathbf{i} \times \mathbf{k} + 3\mathbf{j} \times \mathbf{i} + 5\mathbf{j} \times \mathbf{k} - 3\mathbf{k} \times \mathbf{i} + 4\mathbf{k} \times \mathbf{j} \]
\[ = -4\mathbf{k} - 5\mathbf{j} - 3\mathbf{k} + 5\mathbf{i} - 3\mathbf{j} - 4\mathbf{i} \]
\[ = \mathbf{i} - 8\mathbf{j} - 7\mathbf{k}. \]

Now we will find a point on the line of intersection of the planes. A point of intersection must satisfy both of the equations

\[ x + y - z = 2 \]
\[ 3x - 4y + 5z = 6 \]

and hence must satisfy both

\[ x + y - z = 2 \]
\[ -7y + 8z = 0 \]

and hence both

\[ x + \frac{1}{7}z = 2 \]
\[ -7y + 8z = 0. \]
A point on the line of intersection is thus \((2, 0, 0)\). Symmetric equations for the line of intersection are
\[
\frac{x - 2}{1} = \frac{y}{-8} = \frac{z}{-7}.
\]
The angle, \(\theta\), between the planes satisfies
\[
\cos(\theta) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{-6}{\sqrt{3} \sqrt{50}}
\]
so
\[
\theta = \arccos\left(\frac{-6}{\sqrt{3} \sqrt{50}}\right) \approx 119^\circ.
\]
39. This plane contains the points \((a, 0, 0)\), \((0, b, 0)\), and \((0, 0, c)\). It is thus parallel to both of the vectors \(\mathbf{v}_1 = (-a, b, 0)\) and \(\mathbf{v}_2 = (-a, 0, c)\). Since
\[
\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = bci + acj + abk
\]
(and this is a normal vector for the plane in question), then an equation for the plane in question is
\[
bc(x - a) + ac(y - 0) + ab(z - 0) = 0
\]
or
\[
bcx + acy + abz = abc.
\]
41. The line in question lies in the plane that is parallel to the plane \(x + y + z = 2\) and contains the point \((0, 1, 2)\). This plane has equation
\[
(x - 0) + (y - 1) + (z - 2) = 0
\]
or
\[
x + y + z = 3.
\]
The line in question also lies in the plane that is normal to the vector \(\langle 1, -1, 2 \rangle\) and contains the point \((0, 1, 2)\). This plane has equation
\[
(x - 0) - (y - 1) + 2(z - 2) = 0
\]
or
\[
x - y + 2z = 3.
\]
Two points that lie in both of these planes (and hence on the line of intersection of the planes) are \((3, 0, 0)\) and \((0, 1, 2)\). Thus a direction vector for this line of intersection is \((-3, 1, 2)\). Parametric equations for this line are
\[
x = -3t \\
y = 1 + t \\
z = 2 + 2t.
\]
43. P1 and P3 are parallel but not identical. P2 and P4 are identical.

51. The planes

\[ P_1 : ax + by + cz = -d_1 \]

and

\[ P_2 : ax + by + cz = -d_2 \]

are obviously parallel to each other and both have normal vector \( \mathbf{n} = \langle a, b, c \rangle \). If we let \( A(x_1, y_1, z_1) \) be any point in \( P_1 \) and let \( B(x_2, y_2, z_2) \) be any point in \( P_2 \), then the distance between \( P_1 \) and \( P_2 \) is

\[
D = \left| \text{comp}_\mathbf{n} \overrightarrow{AB} \right| = \frac{|\overrightarrow{AB} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1)|}{|\mathbf{n}|} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.
\]

52. Referring to problem 51, we are looking for a planes \( P_1 \) such that the planes \( P_1 : x + y - 2z = -d_1 \) and \( P_2 : x + 2y - 2z = 1 \) are \( D = 2 \) units apart. This means we must solve

\[
\frac{|d_1 + 1|}{\sqrt{1^2 + 2^2 + (-2)^2}} = 2
\]

or

\[ |d_1 - (-1)| = 6. \]

Since the solutions of this equation are \( d_1 = 5 \) and \( d_1 = -7 \), the two planes in question are \( x + y - 2z = -5 \) and \( x + y - 2z = 7 \).

53. Line \( L_1 : x = y = z \) and line \( L_1 : x + 1 = \frac{y}{2} = \frac{z}{3} \).
have corresponding direction vectors

\[ \mathbf{v}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k} \]

and

\[ \mathbf{v}_2 = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}. \]

Since

\[ \mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \neq \mathbf{0}, \]

we can see that the lines are not parallel. In order for the lines to intersect, we would need to have \((x, y)\) that satisfies both

\[
\begin{align*}
x &= y \\
x + 1 &= \frac{y}{2}
\end{align*}
\]

which would mean \((x, y) = (-2, -2)\). However the equations for \(L_1\) would then give \(z = -2\) and the equations for \(L_2\) would give \(z = -3\). Thus the lines do not intersect and hence are skew. Since the vector \(\mathbf{v}_1 \times \mathbf{v}_2\) is orthogonal to both \(L_1\) and \(L_2\), it is also orthogonal to two parallel planes, \(P_1\) and \(P_2\), where \(P_1\) contains \(L_1\) and \(P_2\) contains \(L_2\). An equation for \(P_1\) is

\[ P_1 : x - 2y + z = 0 \]

and an equation for \(P_2\) is

\[ P_2 : x - 2 + z + 1 = 0. \]

The distance between \(L_1\) and \(L_2\) is the same as the distance between \(P_1\) and \(P_2\) which (by exercise 51) is

\[
D = \frac{\sqrt{1^2 + (-2)^2 + 1^2}}{\sqrt{6}} = \frac{\sqrt{6}}{6}.
\]