Surface Integrals
Surface Integral of a Function

Suppose that \( S \) is a surface in \( \mathbb{R}^3 \) with parametric representation

\[
\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}
\]

\[(u, v) \in \mathcal{D}\]

and suppose that \( f \) is a function of three variables whose domain contains the surface \( S \).

The surface integral of \( f \) over the surface \( S \) is defined as

\[
\iint_S f(x, y, z) \, dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, dA.
\]
Example

Compute the surface integral

\[ \iiint_{S} x^2 \, dS \]

where \( S \) is the unit sphere \( x^2 + y^2 + z^2 = 1 \).
Center of Mass of a Sheet

If a thin sheet has the shape of a surface $S$ and has density function $\rho(x,y,z)$, then the mass of the sheet is

$$m = \iiint_S \rho(x,y,z) \, dS$$

and has center of mass $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{m} \iiint_S x \rho(x,y,z) \, dS$$
$$\bar{y} = \frac{1}{m} \iiint_S y \rho(x,y,z) \, dS$$
$$\bar{z} = \frac{1}{m} \iiint_S z \rho(x,y,z) \, dS.$$
Surface Integrals for Graphs

If $S$ is the surface that is the graph of

\[
\begin{array}{c}
  z = g(x,y) \\
  (x,y) \in D \\
\end{array}
\]

and $f$ is a function of three variables whose domain contains the surface $S$, then the surface integral of $f$ over $S$ is

\[
\iint_S f(x,y,z) \, dS = \iint_D f(x,y,g(x,y)) \sqrt{(z_x)^2 + (z_y)^2 + 1} \, dA.
\]
Example

Evaluate

$$\iint_{S} y^{2} \, dS$$

where $S$ is the graph of

$$z = x + y^{2}$$

$$0 \leq x \leq 1$$

$$0 \leq y \leq 2.$$
Oriented Surfaces

If \( S \) is a smooth orientable surface with parametric representation

\[
\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}
\]

then a unit normal vector to \( S \) at each point \( \mathbf{r}(u, v) \) is

\[
\mathbf{n} = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|} (\mathbf{r}_u \times \mathbf{r}_v).
\]

If \( z = g(x, y) \) is a graph, then a unit normal vector to the graph is

\[
\mathbf{n} = \frac{1}{\sqrt{(z_x)^2 + (z_y)^2 + 1}} (-z_x\mathbf{i} - z_y\mathbf{j} + \mathbf{k}).
\]

For a closed surface, \( S \), we supply \( S \) with the positive orientation by choosing unit normal vectors that point outward from the surface.
Example

Let $S$ be the union of the paraboloid $z = 1 - x^2 - y^2$ and the disk $z = 0, x^2 + y^2 \leq 1$. Choose unit normal vectors that supply $S$ with the positive orientation.

**Solution:** On the disk $z = 0, x^2 + y^2 \leq 1$, we choose the unit normal vectors

$$n = -k.$$

On the paraboloid, we choose the unit normal vectors

$$n = \frac{1}{\sqrt{(z_x)^2 + (z_y)^2 + 1}} (-z_x i - z_y j + k)$$

$$= \frac{1}{\sqrt{4x^2 + 4y^2 + 1}} (2xi + 2yj + k).$$
Example

Let $S$ be the unit sphere $x^2 + y^2 + z^2 = 1$. Choose unit normal vectors that supply $S$ with the positive orientation.

**Solution:** We can write $S$ in parametric form

$$
\mathbf{r}(\theta, \phi) = \cos(\theta) \sin(\phi) \mathbf{i} + \sin(\theta) \sin(\phi) \mathbf{j} + \cos(\phi) \mathbf{k}
$$

$$
0 \leq \theta \leq 2\pi
$$

$$
0 \leq \phi \leq \pi.
$$

Thus

$$
\mathbf{r}_{\theta} = -\sin(\theta) \sin(\phi) \mathbf{i} + \cos(\theta) \sin(\phi) \mathbf{j}
$$

$$
\mathbf{r}_{\phi} = \cos(\theta) \cos(\phi) \mathbf{i} + \sin(\theta) \cos(\phi) \mathbf{j} - \sin(\phi) \mathbf{k}
$$

$$
\mathbf{r}_{\theta} \times \mathbf{r}_{\phi} = -\cos(\theta) \sin^2(\phi) \mathbf{i} - \sin(\theta) \sin^2(\phi) \mathbf{j} - \sin(\phi) \cos(\phi) \mathbf{k}
$$

$$
|\mathbf{r}_{\theta} \times \mathbf{r}_{\phi}| = \sin(\phi).
$$

A unit normal vector is

$$
\mathbf{n} = \frac{1}{|\mathbf{r}_{\theta} \times \mathbf{r}_{\phi}|} (\mathbf{r}_{\theta} \times \mathbf{r}_{\phi}) = -\mathbf{r}(\theta, \phi)
$$

but this unit normal points *inward*. An outward unit normal is

$$
\mathbf{n}(\theta, \phi) = \mathbf{r}(\theta, \phi).
$$
Surface Integral of a Vector Field

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface, $S$, with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over $S$ is defined as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS.$$ 

This integral is also called the flux of $\mathbf{F}$ across $S$. 
Note that if \( S \) is given parametrically by \( \mathbf{r}(u,v) \), then

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} d\mathbf{S}
\]

\[
= \iint_D \mathbf{F} \cdot \left( \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|} (\mathbf{r}_u \times \mathbf{r}_v) \right) |\mathbf{r}_u \times \mathbf{r}_v| dA
\]

\[
= \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA
\]

whereas if \( S \) is given by \( z = g(x,y) \) and \( \mathbf{F} = Pi + Qj + Rk \), then

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} d\mathbf{S}
\]

\[
= \iint_D (Pi + Qj + Rk) \cdot (-z_x i - z_y j + k) dA
\]

\[
= \iint_D (-Pz_x - Qz_y + R) dA
\]
Example

Evaluate
\[ \iiint_F \cdot dS \]
where \( F(x, y, z) = yi + xj + zk \) and \( S \) is the boundary of the region enclosed by the paraboloid \( z = 1 - x^2 - y^2 \) and the disk \( z = 0, x^2 + y^2 \leq 1 \).
Example

Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$. 
Heat Flow

If \( u(x,y,z) \) is the temperature at a point, \( P \), in a body, then the \textit{heat flow} is defined as the vector field

\[
F(x,y,z) = -K \nabla u(x,y,z)
\]

where \( K > 0 \) is a constant called the \textit{conductivity} of the material. The rate of heat flow across a surface, \( S \), in the body is given by the surface integral

\[
\iint_S F \cdot dS = -K \iint_S \nabla u \cdot dS.
\]
Example

The temperature, \( u(x,y,z) \), at a point, \( P \), in a spherical metal ball is proportional to the distance of \( P \) from the center of the ball. Find the rate of heat flow across a sphere of radius \( a \) with center at the center of the ball.