Direct Sums of Subspaces and Fundamental Subspaces

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1 Direct Sums

Suppose that $V$ is a vector space and that $H$ and $K$ are subspaces of $V$ such that $H \cap K = \{0\}$. The **direct sum** of $H$ and $K$ is the set of vectors

$$H \oplus K = \{u + v \mid u \in H \text{ and } v \in K\}.$$

**Example 1** In $V_2$, the subspaces $H = \text{Span}(e_1)$ and $K = \text{Span}(e_2)$ satisfy $H \cap K = \{0_2\}$ and we observe that

$$H \oplus K = \{u + v \mid u \in \text{Span}(e_1) \text{ and } v \in \text{Span}(e_2)\}$$

is actually the set of all possible linear combinations of the standard basis vectors $e_1$ and $e_2$. Thus $H \oplus K = V_2$.

**Example 2** Let $P_2(R)$ be the vector space of all polynomial functions with domain $R$ and let $H$ and $K$ be the subspaces $H = \text{Span}(p_1)$ and $K = \text{Span}(p_2)$ where

- $p_1(t) = t$ for all $t \in R$
- $p_2(t) = t^2$ for all $t \in R$.

Note that $H \cap K = \{z\}$ because all polynomials in $H$ have the form $p(t) = ct$ (where $c$ can be any scalar) and all polynomials in $K$ have the form $p(t) = kt^2$ (where $k$ can be any scalar) and in order to have $p \in H \cap K$, it must be true that $ct = kt^2$ for all $t \in R$. This implies that $c(1) = k(1)^2$ and hence that
c = k and it must also be true that c(2) = c(2)^2 and hence c = k = 0. In this case, H ⊕ K is the set of all polynomials, p, of the form p(t) = kt^2 + ct where k and c can be any scalars. Another way to say this is that H ⊕ K = Span(p_1, p_2).

**Exercise 3** Let F(R) be the vector space of all functions with domain R and let H be the set of all **even** functions in F(R) and let K be the set of all **odd** functions in F(R). (Recall that an even function is a function, f, for which f(-x) = f(x) for all x ∈ R, and an odd function is a function, f, for which f(-x) = -f(x) for all x ∈ R.)

1. Prove that H and K are both subspaces of F(R).
2. Prove that H ∩ K = {z}.  
3. Prove that H ⊕ K = F(R). **Hint:** For any function f ∈ F(R), prove that f_1 ∈ H and f_2 ∈ K where

\[
\begin{align*}
f_1(x) &= \frac{1}{2}(f(x) + f(-x)) \\
f_2(x) &= \frac{1}{2}(f(x) - f(-x))
\end{align*}
\]

and then observe that f = f_1 + f_2.

4. As an illustration of this result, let f ∈ F(R) be the function f(x) = e^x and find an even function, f_1, and an odd function, f_2, such that f = f_1 + f_2.

Some basic facts about direct sums are given in the following theorem.

**Theorem 4 (Basic Facts About Direct Sums)** Suppose that V is a vector space and that H and K are subspaces of V such that H ∩ K = {0}.

1. H ⊕ K is a subspace of V.
2. Each vector v ∈ H ⊕ K can be written uniquely as a sum of vectors, v_1 + v_2, where v_1 ∈ H and v_2 ∈ K.
3. If B_1 = \{u_1, u_2, \ldots, u_p\} is a basis for H and B_2 = \{v_1, v_2, \ldots, v_q\} is a basis for K, then B_1 ∩ B_2 = ∅ and B_1 ∪ B_2 is a basis for H ⊕ K.
4. \( \dim (H \oplus K) = \dim (H) + \dim (K) \).

**Proof.** (Proof of statement 1:) In order to prove statement 1, we must show that \( H \oplus K \) is closed under addition and scalar multiplication.

Let \( u \) and \( v \) be vectors in \( H \oplus K \). Since \( u \in H \oplus K \), then \( u = u_1 + u_2 \) for some vectors \( u_1 \in H \) and \( u_2 \in K \). Likewise, since \( v \in H \oplus K \), then \( v = v_1 + v_2 \) for some vectors \( v_1 \in H \) and \( v_2 \in K \). Thus

\[
 u + v = (u_1 + u_2) + (v_1 + v_2) = (u_1 + v_1) + (u_2 + v_2).
\]

Since \( H \) is a subspace of \( V \) (and hence closed under addition) we know that \( u_1 + v_1 \in H \). Likewise, \( u_2 + v_2 \in K \). Thus we see (from the above equation) that \( u + v \in H \oplus K \). This shows that \( H \oplus K \) is closed under addition.

Let \( u \) be a vector in \( H \oplus K \) and let \( c \) be a scalar. Since \( u \in H \oplus K \), then \( u = u_1 + u_2 \) for some vectors \( u_1 \in H \) and \( u_2 \in K \). Thus

\[
 cu = c(u_1 + u_2) = cu_1 + cu_2.
\]

Since \( H \) is a subspace of \( V \) (and hence closed under scalar multiplication) we know that \( cu_1 \in H \). Likewise, \( cu_2 \in K \). Thus we see (from the above equation) that \( cu \in H \oplus K \). This shows that \( H \oplus K \) is closed under scalar multiplication. Therefore \( H \oplus K \) is a subspace of \( V \).

(Proof of statement 2:) Let \( v \) be any vector in \( H \oplus K \). Then \( v = v_1 + v_2 \) for some vectors \( v_1 \in H \) and \( v_2 \in K \). Suppose also that there exist vectors \( u_1 \in H \) and \( u_2 \in K \) such that \( v = u_1 + u_2 \). Then

\[
 v_1 + v_2 = u_1 + u_2
\]

which implies that

\[
 v_1 + (-u_1) = u_2 + (-v_2).
\]

Note that the vector \(-u_1\) (which is the additive inverse of \( u_1 \)) is in \( H \) (because \( H \) is a subspace of \( V \) and \( u_1 \in H \)). Also, since \( v_1 \in H \), we see that \( v_1 + (-u_1) \in H \) (since \( H \) is closed under addition). By similar reasoning, we observe that \( u_2 + (-v_2) \in K \). However, since \( v_1 + (-u_1) = u_2 + (-v_2) \), we see that \( v_1 + (-u_1) \in H \cap K \). Since \( H \cap K = \{0\} \), we conclude that \( v_1 + (-u_1) = 0 \) and hence that \( v_1 = u_1 \) and by the same reasoning that \( v_2 = u_2 \). This shows that there is only one way to write \( v \) as a sum of vectors from \( H \) and \( K \) (which is what we wanted to prove).
(Proof of statement 3): Let $B_1 = \{u_1, u_2, \ldots, u_p\}$ be a basis for $H$ and $B_2 = \{v_1, v_2, \ldots, v_q\}$ be a basis for $K$. Since $0 \notin B_1$ (because $B_1$ is a basis) and, likewise, $0 \notin B_2$, and $B_1 \subseteq H$ and $B_2 \subseteq K$, and $H \cap K = \{0\}$, we see that $B_1 \cap B_2 = \emptyset$.

To show that $B_1 \cup B_2$ is a basis for $H \oplus K$, we must show that $B_1 \cup B_2$ is linearly independent and spans $H \oplus K$. We will first show that $B_1 \cup B_2$ is linearly independent. Suppose that there exist scalars $c_1, c_2, \ldots, c_r$ and $d_1, d_2, \ldots, d_q$ such that

$$c_1 u_1 + c_2 u_2 + \cdots + c_r u_r + d_1 v_1 + d_2 v_2 + \cdots + d_q v_q = 0.$$  

Since 

$$a = c_1 u_1 + c_2 u_2 + \cdots + c_r u_r \in H$$  

and 

$$b = d_1 v_1 + d_2 v_2 + \cdots + d_q v_q \in K,$$  

and $a + b = 0$, we conclude that $a = 0$ and $b = 0$. This is because there is only one way to write $0$ as a sum of vectors from $H$ and $K$ (by statement 2 of this theorem) and clearly this one way is $0 + 0 = 0$. Thus

$$c_1 u_1 + c_2 u_2 + \cdots + c_r u_r = 0$$  

and 

$$d_1 v_1 + d_2 v_2 + \cdots + d_q v_q = 0.$$  

Since $B_1$ is a basis for $H$ (and is hence a linearly independent set), we conclude that $c_1 = c_2 = \cdots = c_r = 0$. Likewise we conclude that $d_1 = d_2 = \cdots = d_q = 0$. This proves that $B_1 \cup B_2$ is a linearly independent set.

We now show that $B_1 \cup B_2$ spans $H \oplus K$. Let $v$ be any vector in $H \oplus K$. Then $v = v_1 + v_2$ for some $v_1 \in H$ and $v_2 \in K$. However, since $B_1$ is a basis for $H$, then $v_1$ is a linear combination of the vectors in $B_1$. Likewise $v_2$ is a linear combination of the vectors in $B_2$. Thus $v = v_1 + v_2$ is a linear combination of the vectors in $B_1 \cup B_2$, which shows that $B_1 \cup B_2$ spans $H \oplus K$.

In conclusion, $B_1 \cup B_2$ is a basis for $H \oplus K$.

(Proof of statement 4): From statement 3, we know that if $B_1 = \{u_1, u_2, \ldots, u_p\}$ is a basis for $H$ and $B_2 = \{v_1, v_2, \ldots, v_q\}$ is a basis for $K$, then $B_1 \cup B_2$ is a basis for $H \oplus K$ that contains exactly $p + q$ vectors (because $B_1 \cap B_2 = \emptyset$). Therefore

$$\dim (H \oplus K) = p + q = \dim (H) + \dim (K).$$  

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If $V$ is a vector space and $H$ and $K$ are subspaces of $V$ (with $H \cap K = \{0\}$) and such that $H \oplus K = V$, then we say that that $H$ and $K$ are complementary to each other. Thus, to say that $H$ and $K$ are complementary to each other means that we can “build” the entire vector space $V$ by forming the direct sum of $H$ and $K$. The use of the term “complementary” reminds us of complementary angles in geometry. (Two angles are said to be complementary to each other if their sum is $90^\circ$.)

**Example 5** In Example 1, $H$ and $K$ are complementary subspaces of $V_2$ because $H \oplus K = V_2$. In Example 2, $H$ and $K$ are not complementary subspaces of $P_2(R)$ because $H \oplus K \neq P_2(R)$. In Exercise 3, $H$ and $K$ are complementary subspaces of $F(R)$ because $H \oplus K = F(R)$.

### 2 Orthogonality and Orthogonal Complements

If $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ are two vectors in $V_n$, then

$u^T v = [u_1v_1 + u_2v_2 + \cdots + u_nv_n] = v^T u$.

(Note that this is a $1 \times 1$ matrix.)

If $u^T v = [0]$, then we say that $u$ and $v$ are orthogonal to each other. The word “orthogonal” means “perpendicular” and the reason that we use this word is that, in $V_2$ or $V_3$, if $u^T v = [0]$, then there is actually an angle of $90^\circ$ between the vectors $u$ and $v$. This geometric idea is discussed in Calculus. Those who have taken Calculus III will recognize that $u^T v$ is actually the dot product of $u$ and $v$.

Two subspaces, $H$ and $K$, of $V_n$ are said to be orthogonal to each other if $u^T v = [0]$ for all $u \in H$ and $v \in K$.

**Example 6** In $V_2$, the subspaces $H = \text{Span} (e_1)$ and $K = \text{Span} (e_2)$ are orthogonal to each other. This is because any vector $u \in H$ has the form

$$u = \begin{bmatrix} t \\ 0 \end{bmatrix}$$
and any vector $v \in K$ has the form

$$v = \begin{bmatrix} 0 \\ s \end{bmatrix}$$

and we easily observe that

$$u^T v = \begin{bmatrix} t & 0 \end{bmatrix} \begin{bmatrix} 0 \\ s \end{bmatrix} = [0].$$

The geometric view of this, of course, is that $H$ can be viewed as the $x$ axis in $\mathbb{R}^2$ and $K$ can be viewed as the $y$ axis in $\mathbb{R}^2$ and, clearly, the $x$ axis and $y$ axis are perpendicular to each other.

**Exercise 7**

1. In $V_2$, let $H = \text{Span} (u_1)$ and $K = \text{Span} (u_2)$ where

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   (a) Explain (algebraically) why the subspaces $H$ and $K$ are not orthogonal to each other.

   (b) Explain (geometrically) why the subspaces $H$ and $K$ are not orthogonal to each other. (Draw pictures of $H$ and $K$.)

   (c) Are $H$ and $K$ complementary subspaces of $V_2$? Explain.

2. In $V_2$, let $H = \text{Span} (u_1)$ and $K = \text{Span} (u_2)$ where

$$u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

   (a) Explain (algebraically) why the subspaces $H$ and $K$ are orthogonal to each other.

   (b) Explain (geometrically) why the subspaces $H$ and $K$ are orthogonal to each other. (Draw pictures of $H$ and $K$.)

   (c) Are $H$ and $K$ complementary subspaces of $V_2$? Explain.

3. In $V_3$, let $H = \text{Span} (u_1, u_2)$ and $K = \text{Span} (u_3)$ where

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix}, \quad \text{and} \quad u_3 = \begin{bmatrix} 15 \\ 3 \\ -7 \end{bmatrix}.$$
(a) Explain (algebraically) why the subspaces \(H\) and \(K\) are orthogonal to each other.

(b) Explain (geometrically) why the subspaces \(H\) and \(K\) are orthogonal to each other. (What type of geometric object is \(H\)? What type of geometric object is \(K\)?)

(c) Are \(H\) and \(K\) complementary subspaces of \(V_3\)? Explain.

**Lemma 8** If \(H\) and \(K\) are subspaces of \(V_n\) that are orthogonal to each other, then \(H \cap K = \{0\}\).

**Proof.** Suppose that \(H\) and \(K\) are subspaces of \(V_n\) that are orthogonal to each other. Let

\[
v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in H \cap K.
\]

Then \(v\) is orthogonal to itself, so

\[
v^T v = [v_1^2 + v_2^2 + \cdots + v_n^2] = [0].
\]

Since \(v_1^2 + v_2^2 + \cdots + v_n^2 = 0\), we conclude that \(v = 0\). We have thus proved that \(H \cap K \subseteq \{0\}\). The fact that \(\{0\} \subseteq H \cap K\) is obvious. Therefore \(H \cap K = \{0\}\).

The fact that \(H \cap K = \{0\}\) when \(H\) and \(K\) are orthogonal to each other means that \(H \oplus K\) is defined when \(H\) and \(K\) are orthogonal to each other. The following theorem (which we state without proof) says that any subspace, \(H\), of \(V_n\) has an orthogonal complement – that is, a complementary subspace that is orthogonal to \(H\).

**Theorem 9 (The Orthogonal Decomposition Theorem)** If \(H\) is any subspace of \(V_n\), then there is a subspace, \(H^\perp\), of \(V_n\) such that \(H\) and \(H^\perp\) are orthogonal to each other and complementary to each other.
3 Orthogonal Decompositions Induced by Matrices

Recall that if \( A \) is an \( m \times n \) matrix, then \( A \) can be written as

\[
A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
\]

where \( a_1, a_2, \ldots, a_n \) are the column vectors of \( A \) (which are each \( m \times 1 \) matrices) and \( b_1, b_2, \ldots, b_m \) are the row vectors of \( A \) (which are each \( 1 \times n \) matrices). The following four subspaces (two of which are subspaces of \( V_n \) and the other two of which are subspaces of \( V_m \)) are called the fundamental subspaces of the matrix \( A \):

1. The row space of \( A \),

\[
\text{row} (A) = \text{Span} \left( b_1^T, b_2^T, \ldots, b_m^T \right)
\]

is a subspace of \( V_n \).

2. The null space of \( A \),

\[
\text{Nul} (A) = \{ x \in V_n \mid A x = 0_m \}
\]

is a subspace of \( V_n \).

3. The column space of \( A \),

\[
\text{Col} (A) = \text{Span} \left( a_1, a_2, \ldots, a_n \right)
\]

is a subspace of \( V_m \).

4. The null space of \( A^T \),

\[
\text{Nul} (A^T) = \{ x \in V_m \mid A^T x = 0_n \}
\]

is a subspace of \( V_m \).
The following theorem is often referred to as the Fundamental Theorem of Linear Algebra.

**Theorem 10 (Fundamental Theorem of Linear Algebra)** If $A$ is an $m \times n$ matrix, then $\text{row } (A)$ and $\text{Nul } (A)$ are orthogonal complements of each other in $V_n$ and $\text{Col } (A)$ and $\text{Nul } (A^T)$ are orthogonal complements of each other in $V_m$.

**Proof.** First we prove that $\text{row } (A)$ and $\text{Nul } (A)$ are orthogonal complements in $V_n$. Let $v \in \text{row } (A)$ and $w \in \text{Nul } (A)$. Then

$$v = c_1b_1^T + c_2b_2^T + \cdots + c_mb_m^T$$

for some scalars $c_1, c_2, \ldots, c_m$ and

$$Aw = 0_m.$$  

Since

$$A = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

we see that

$$0_m = Aw = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}w = \begin{bmatrix} b_1w \\ b_2w \\ \vdots \\ b_mw \end{bmatrix}$$

and thus $b_jw = [0]$ for all $j = 1, 2, \ldots, m$. Since $b_jw = w^Tb_j^T$ for all $j = 1, 2, \ldots, m$, we see that $w^Tb_j^T = [0]$ for all $j = 1, 2, \ldots, m$. Thus

$$w^Tv = w^T(c_1b_1^T + c_2b_2^T + \cdots + c_mb_m^T) = c_1w^Tb_1^T + c_2w^Tb_2^T + \cdots + c_mw^Tb_m^T = [0] + [0] + \cdots + [0] = [0]$$

which means that $w$ and $v$ are orthogonal to each other. We have now proved that $\text{row } (A)$ and $\text{Nul } (A)$ are orthogonal to each other.
Since
\[ \dim \left( \text{row} \left( A \right) \right) = \dim \left( \text{Col} \left( A \right) \right) = \text{number of pivot columns in } A \]
and
\[ \dim \left( \text{Nul} \left( A \right) \right) = \text{number of non–pivot columns in } A, \]
then
\[ \dim \left( \text{row} \left( A \right) \right) + \dim \left( \text{Nul} \left( A \right) \right) = n. \]
By part 4 of Theorem 4, we see thus see that
\[ \dim \left( \text{row} \left( A \right) \oplus \text{Nul} \left( A \right) \right) = n \]
and hence that \( \text{row} \left( A \right) \oplus \text{Nul} \left( A \right) = V_n \). This completes the proof that \( \text{row} \left( A \right) \) and \( \text{Nul} \left( A \right) \) are orthogonal complements of each other.

Since \( \text{Col} \left( A \right) = \text{row} \left( A^T \right) \), the fact that \( \text{Col} \left( A \right) \) and \( \text{Nul} \left( A^T \right) \) are orthogonal complements of each other follows immediately from what we have already proved.

We now provide an example that illustrates the fundamental theorem.

**Example 11** Let us find the fundamental subspaces of the matrix

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
-1 & 5 & 0
\end{bmatrix}
\]

Since
\[
\text{rref} \left( A \right) = \begin{bmatrix}
1 & 0 & \frac{15}{7} \\
0 & 1 & \frac{3}{7} \\
0 & 0 & 0
\end{bmatrix},
\]
we see that the pivot columns of \( A \) are columns 1 and 2 and that
\[
\text{Col} \left( A \right) = \text{Span} \left( \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \right)
\]
and that \( \dim \left( \text{Col} \left( A \right) \right) = 2 \). In addition, we see that
\[
\text{Nul} \left( A \right) = \text{Span} \left( \begin{bmatrix} -\frac{15}{7} \\ -\frac{3}{7} \\ 1 \end{bmatrix} \right) = \text{Span} \left( \begin{bmatrix} 15 \\ 3 \\ -7 \end{bmatrix} \right)
\]

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and that \( \dim(\text{Nul}(A)) = 1 \). We also can see from this that

\[
\text{row}(A) = \text{Span} \left( \begin{bmatrix} 1 \\ \frac{15}{7} \\ \frac{1}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \end{bmatrix} \right) = \text{Span} \left( \begin{bmatrix} 7 \\ 0 \\ 15 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} \right).
\]

Observe that \( \text{row}(A) \) and \( \text{Nul}(A) \) are orthogonal to each other because the vector

\[
\begin{bmatrix} 15 \\ 3 \\ -7 \end{bmatrix},
\]

which is a basis for \( \text{Nul}(A) \) is orthogonal to each vector in the basis

\[
\left\{ \begin{bmatrix} 7 \\ 0 \\ 15 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} \right\}
\]

of \( \text{row}(A) \).

Now note that

\[
A^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 5 \\ 3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref}(A^T)
\]

which shows that

\[
\text{Nul}(A^T) = \text{Span} \left( \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right)
\]

and observe that \( \text{Nul}(A^T) \) is orthogonal to \( \text{Col}(A) \) because the vector

\[
\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}
\]

which is a basis for \( \text{Nul}(A^T) \), is orthogonal to each of the vectors in the basis

\[
\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \right\}
\]
of Col \((A)\). In addition, since row \((A) = Col \((A^T)\)\), we see that another basis for row \((A)\) (which is different from the one obtained earlier) is

\[
\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix} \right\}
\]

(because the pivot columns of \(A^T\) are columns 1 and 3).

Finally, observe that row \((A) \oplus Nul \((A)\) = \(V_3\) because

\[
\left\{ \begin{bmatrix} 7 \\ 0 \\ 15 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 15 \\ 3 \\ -7 \end{bmatrix} \right\}
\]

is a linearly independent set of vectors in \(V_3\) and also observe that Col \((A) \oplus Nul \((A^T)\) = \(V_3\) because

\[
\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}
\]

is a linearly independent set of vectors that spans \(V_3\).

**Exercise 12** Find the four fundamental subspaces of the following matrices and verify that row \((A)\) and \(Nul \((A)\) are orthogonal complements of each other and that Col \((A)\) and \(Nul \((A^T)\) are orthogonal complements of each other.

1. \(A = \begin{bmatrix} -4 \\ 7 \\ 8 \end{bmatrix}\)

2. \(A = \begin{bmatrix} 10 & -6 \\ -8 & -5 \\ 7 & 6 \end{bmatrix}\)

3. \(A = \begin{bmatrix} -6 & 0 \\ 5 & -10 \end{bmatrix}\)
4. \[ A = \begin{bmatrix} 1 & 1 \end{bmatrix} \]

5. \[ A = \begin{bmatrix} -3 & -10 & 5 & -4 \\ -8 & -1 & -10 & 2 \\ 3 & -1 & -9 & 8 \end{bmatrix} \]