1.

\[ V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0 \right\}. \]

a. Suppose that \( u \) and \( v \) are in \( V \). Then

\[ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \]

where \( u_1 \geq 0 \) and \( u_2 \geq 0 \) and

\[ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \]

where \( v_1 \geq 0 \) and \( v_2 \geq 0 \).

This implies that \( u + v \) is in \( V \) because

\[ u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} \]

and it is clearly true that \( u_1 + v_1 \geq 0 \) and \( u_2 + v_2 \geq 0 \).

b. Let

\[ u = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \]

and let \( c = -3 \). Then

\[ cu = -3 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ -15 \end{bmatrix} \]

is not in \( V \). (Note that part a shows that \( V \) is closed under addition and part b shows that \( V \) is not closed under scalar multiplication.)

3.

\[ H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x^2 + y^2 \leq 1 \right\}. \]

If we let

\[ u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

and \( c = 2 \), then \( u \in H \) but \( cu \notin H \). Thus, \( H \) is not closed under scalar multiplication and so \( H \) is not a subspace of \( \mathbb{R}^2 \).
5. Note that $P_n$ is the vector space of all polynomial functions of degree at most $n$. Thus, $P_n$ is the set of all functions $p$ (with domain $(-\infty, \infty)$) of the form

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_2 t^2 + a_1 t + a_0$$

where the coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ can be any real constants.

The set of all polynomial functions of the form $p(t) = at^2$ is a subspace of $P_n$.

7. The set of all polynomial functions with degree at most $n$ and with integer coefficients is not a subspace of $P_n$ because it is not closed under scalar multiplication. For example, the function $p(t) = 3t^2 - 4t + 12$ is in this set (considered as a subset of $P_2$), but the function $0.5p(t) = 1.5t^2 - 2t + 6$ is not.

9. If $H$ is the set of all vectors in $\mathbb{R}^3$ that have the form

$$\begin{bmatrix}
  s \\
  3s \\
  2s
\end{bmatrix},$$

then

$$H = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right\}$$

and thus $H$ is a subspace of $\mathbb{R}^3$.

11. If $H$ is the set of all vectors in $\mathbb{R}^3$ that have the form

$$\begin{bmatrix}
  5b + 2c \\
  b \\
  c
\end{bmatrix},$$

then

$$H = \text{Span} \left\{ \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and thus $H$ is a subspace of $\mathbb{R}^3$.

13. a. $\mathbf{w} \notin \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$. There are only three vectors in the set $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$. They are the vectors $\mathbf{v}_1, \mathbf{v}_2, \text{ and } \mathbf{v}_3$ themselves.

b. There are infinitely many vectors in the set $\text{Span} \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$. This set consists of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \text{ and } \mathbf{v}_3$.

c. 
shows that $w \in \text{Span}\{v_1, v_2, v_3\}$. For example, $w$ can be expressed as $w = v_1 + v_2$.

15. The set of all vectors of the form

$$\begin{bmatrix} 3a + b \\ 4 \\ a - 5b \end{bmatrix}$$

is not a vector space because, for example, the vectors

$$\begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

are in this set, but the sum of these two vectors,

$$\begin{bmatrix} 4 \\ 8 \\ -4 \end{bmatrix},$$

is not in this set.

17. The set of all vectors of the form

$$\begin{bmatrix} a - b \\ b - c \\ c - a \\ b \end{bmatrix}$$

is a vector space. In fact, this set of vectors is precisely

$$\text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$
a. The zero vector (which is really the zero function in this case) is in the described set because
\[ 0 = 0 \cos(\omega t) + 0 \sin(\omega t). \]

b. The set is closed under addition because if we take any two vectors
\[ y_1(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \]
and
\[ y_2(t) = c_3 \cos(\omega t) + c_4 \sin(\omega t) \]
in this set, then the sum
\[
(y_1 + y_2)(t) \\
= (c_1 \cos(\omega t) + c_2 \sin(\omega t)) + (c_3 \cos(\omega t) + c_4 \sin(\omega t)) \\
= (c_1 + c_3) \cos(\omega t) + (c_2 + c_4) \sin(\omega t)
\]
is also in this set.

c. This set is closed under scalar multiplication because if we take any vector
\[ y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \]
in this set and any scalar \( c \), then the vector
\[
(cy)(t) \\
= c(c_1 \cos(\omega t) + c_2 \sin(\omega t)) \\
= (cc_1) \cos(\omega t) + (cc_2) \sin(\omega t)
\]
is also in this set.

21. The set of all matrices of the form
\[
\begin{bmatrix}
  a & b \\
  0 & d
\end{bmatrix}
\]
is a subspace of \( M_{2 \times 2} \) because

a. 
\[
0 = \begin{bmatrix}
  0 & 0 \\
  0 & 0
\end{bmatrix},
\]
is in this set.

b. If we take two vectors
\[
M_1 = \begin{bmatrix}
  a_1 & b_1 \\
  0 & d_1
\end{bmatrix}
\]
and
\[
M_2 = \begin{bmatrix}
  a_2 & b_2 \\
  0 & d_2
\end{bmatrix}
\]
in this set, then the sum
\[ M_1 + M_1 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 + 0 & d_1 + d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & d_1 + d_2 \end{bmatrix} \]
is in this set (so the set is closed under addition).

c. If we take a vector
\[
M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}
\]
in this set and any scalar \( c \), the vector
\[
cM = \begin{bmatrix} c \cdot a & c \cdot b \\ c \cdot 0 & c \cdot d \end{bmatrix} = \begin{bmatrix} ca & cb \\ 0 & cd \end{bmatrix}
\]
is in this set (so the set is closed under scalar multiplication).

23.

a. This statement is not really clear. If the statement is saying that \( f(t) = 0 \) for some particular value of \( t \), then \( f \) is the zero vector in \( V \), then the statement is false. However, if the statement is saying that \( f(t) = 0 \) for all \( t \in \mathbb{R} \), then \( f \) is the zero vector in \( V \), then the statement is true.

b. I guess the intention is that this statement is false. The set of all arrows in three dimensional space is an example of a vector space, but in general, a vector can be something much more general (such as a matrix or a function).

c. False. The set \( H \) must also be closed under addition and scalar multiplication.

d. True.

e. True.

24.

a. True.

b. True.

c. True. (It is a subspace of itself.)

d. False. \( \mathbb{R}^2 \) is not even a subset of \( \mathbb{R}^3 \). However, the set of all vectors of the form
\[
\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}
\]
is a subspace of \( \mathbb{R}^3 \) that is really “just like” \( \mathbb{R}^2 \).

e. False (or actually just worded in a way that does not make sense). The correct wording is: A subset \( H \) of a vector space \( V \) is a subspace of \( V \) if the following conditions are satisfied:

i. The zero vector of \( V \) is in \( H \).
ii. For every pair of vectors \( u \) and \( v \) in \( H \), the vector \( u + v \) is also in \( H \).

iii. For every vector \( u \) in \( H \) and for every scalar \( c \), the vector \( cu \) is in \( H \).

26. **Proof that for any \( u \in V \), the vector \(-u\) is the unique vector in \( V \) such that \( u + (-u) = 0 \).**

Let \( u \in V \) be given and suppose that \( w \) satisfies \( u + w = 0 \). Then

\[
\begin{align*}
-\u + (\u + \w) &= -\u + 0 & \text{by adding } -\u \text{ to both sides} \\
(-\u + \u) + \w &= -\u + 0 & \text{by associativity of addition} \\
0 + \w &= -\u + 0 & \text{property of additive inverses} \\
\w + 0 &= -\u + 0 & \text{commutative property of addition} \\
\w &= -\u & \text{by the property of the zero vector}
\end{align*}
\]

27. **done in class.**

28. **Proof that for every scalar \( c \), we have \( c0 = 0 \).**

For any given scalar \( c \), we have

\[
\begin{align*}
\c0 &= \c(0 + 0) & \text{additive property of the zero vector} \\
\c0 &= \c0 + \c0 & \text{distribution of scalar multiplication over addition} \\
\c0 + (-\c0) &= (\c0 + \c0) + (-\c0) & \text{adding } -\c0 \text{ to both sides} \\
0 &= \c0 + (\c0 + (-\c0)) & \text{property of additive inverses} \\
0 &= \c0 + 0 & \text{associativity of addition} \\
0 &= \c0 + 0 & \text{property of additive inverses} \\
0 &= \c0 & \text{additive property of the zero vector}
\end{align*}
\]

29. **Proof that for every \( u \in V \), we have \(-1u = -u\).**

Let \( u \) be a vector in \( V \). Then

\[
\begin{align*}
\u + (-1\u) &= 1\u + (-1\u) & \text{using the axiom that } 1\u = \u \\
\u + (-1\u) &= (1 + (-1))\u & \text{distribution of scalar mult. over reg. add.} \\
\u + (-1\u) &= 0\u & \text{using the fact that } 1 + (-1) = 0 \\
\u + (-1\u) &= 0 & \text{using the fact that } 0\u = \u
\end{align*}
\]

Since \( \u + (-1\u) = 0 \), it must be the case (by Exercise 26) that \(-1\u = -\u\).

30. **If \( u \) is a vector in some vector space \( V \) and \( c \) is a non–zero scalar such that \( cu = 0 \), then**
\[
\frac{1}{c} (cu) = \frac{1}{c} \cdot 0 \quad \text{multiplying both sides by } \frac{1}{c}
\]
\[
(\frac{1}{c}c)u = \frac{1}{c} \cdot 0 \quad \text{associativity}
\]
\[
1u = \frac{1}{c} \cdot 0 \quad \text{the fact that } \frac{1}{c}c = 1
\]
\[
u = \frac{1}{c} \cdot 0 \quad \text{the axiom that } 1u = u
\]
\[
u = 0 \quad \text{the result of Exercise 28}
\]

31. Suppose that \( H \) is a subspace of a vector space \( V \) and suppose that \( H \) contains two particular vectors, \( u \) and \( v \). Since \( H \) is closed under addition and scalar multiplication, then \( H \) must also contain all vectors of the form \( c_1u + c_2v \) (for any scalars \( c_1 \) and \( c_2 \)). This means that \( H \) must in fact contain Span\{\( u, v \)\}.

32. We are given that \( H \) and \( K \) are both subspaces of a vector space \( V \) and we want to prove that \( H \cap K \) (which is by definition the set of all vectors that belong to both \( H \) and \( K \)) is also a subspace of \( V \).

a. Since \( 0 \in H \) and \( 0 \in K \), then \( 0 \in H \cap K \).

b. Suppose that \( u \in H \cap K \) and \( v \in H \cap K \). Since \( u \) and \( v \) are both in \( H \), then \( u + v \) is also in \( H \). In addition, since \( u \) and \( v \) are both in \( K \), then \( u + v \) is also in \( K \). This means that \( u + v \in H \cap K \).

c. Suppose that \( u \in H \cap K \) and that \( c \) is a scalar. Since \( u \in H \), then \( cu \) is also in \( H \). Also, since \( u \in K \), then \( cu \) is also in \( K \). This means that \( cu \in H \cap K \).

Note that if \( H \) and \( K \) are subspaces of \( V \), then it is generally not true that \( H \cup K \) (which is the set of all vectors that are in either \( H \) or \( K \)) is a subspace of \( V \). For example, in \( \mathbb{R}^2 \), we could take \( H \) to be any line through the origin and take \( K \) to be any other (different) line through the origin. Then \( H \) and \( K \) are both subspaces of \( \mathbb{R}^2 \) but \( H \cup K \) is not a subspace of \( \mathbb{R}^2 \) because if we take the sum of a non–zero vector in \( H \) and a non–zero vector in \( K \), then we get a non–zero vector which is neither in \( H \) nor in \( K \). Thus, \( H \cup K \) is not closed under addition. (Specific example: if \( H \) is the horizontal axis and \( K \) is the vertical axis, then \( u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in H \cup K \) and \( v = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in H \cup K \), but \( u + v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin H \cup K \).)