1. \[ A \mathbf{w} = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]
so \( \mathbf{w} \in \text{Nul}(A) \).

3. Using \[ \begin{bmatrix} 1 & 3 & 5 & 0 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix}, \]
we see that all vectors \( \mathbf{x} \) satisfying \( A\mathbf{x} = \mathbf{0} \) have the form
\[ \mathbf{x} = x_3 \begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix}. \]
Thus, \( \text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix} \right\} \).

5. Using \[ \begin{bmatrix} 1 & -2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \]
we see that all vectors \( \mathbf{x} \) satisfying \( A\mathbf{x} = \mathbf{0} \) have the form
\[ \mathbf{x} = x_2 \begin{bmatrix} 2 \\ 9 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 9 \\ 1 \\ 0 \end{bmatrix}. \]
Thus
7. The set described is not a subspace of $\mathbb{R}^3$ because it does not contain the zero vector.

9. The set described is the set of all vectors

$$
\begin{bmatrix}
    a \\
    b \\
    c \\
    d
\end{bmatrix} \in \mathbb{R}^4
$$

such that

$$
\begin{bmatrix}
    1 & -2 & -4 & 0 \\
    2 & 0 & -1 & -3
\end{bmatrix}
\begin{bmatrix}
    a \\
    b \\
    c \\
    d
\end{bmatrix} =
\begin{bmatrix}
    0 \\
    0
\end{bmatrix}.
$$

Thus this set is the null space of the matrix

$$
A = \begin{bmatrix}
1 & -2 & -4 & 0 \\
2 & 0 & -1 & -3
\end{bmatrix}
$$

and is hence a subspace of $\mathbb{R}^4$.

11. The set described is not a subspace of $\mathbb{R}^4$ because it does not contain the zero vector. In order to make the second component of such a vector equal zero, we would have to choose $d = -5$, but that would make the fourth component equal to $-5$.

13. The set described is the set of all vectors in $\mathbb{R}^3$ that have the form

$$
\begin{bmatrix}
    c - 6d \\
    d \\
    c
\end{bmatrix} = c
\begin{bmatrix}
    1 \\
    0 \\
    1
\end{bmatrix} + d
\begin{bmatrix}
    -6 \\
    1 \\
    0
\end{bmatrix} =
\begin{bmatrix}
    1 & -6 \\
    0 & 1 \\
    1 & 0
\end{bmatrix}
\begin{bmatrix}
    c \\
    d
\end{bmatrix}.
$$

This set is thus the column space of the matrix

$$
A = \begin{bmatrix}
1 & -6 \\
0 & 1 \\
1 & 0
\end{bmatrix}
$$
15. The set described is the column space of the matrix

\[
A = \begin{bmatrix}
0 & 2 & 3 \\
1 & 1 & -2 \\
4 & 1 & 0 \\
3 & -1 & -1
\end{bmatrix}
\]

17. \( \text{Col}(A) \) is a subspace of \( \mathbb{R}^4 \) and \( \text{Nul}(A) \) is a subspace of \( \mathbb{R}^3 \).

19. \( \text{Col}(A) \) is a subspace of \( \mathbb{R}^2 \) and \( \text{Nul}(A) \) is a subspace of \( \mathbb{R}^5 \).

21. For

\[
A = \begin{bmatrix}
2 & -6 \\
-1 & 3 \\
-4 & 12 \\
3 & -9
\end{bmatrix}
\]

a non–zero vector in \( \text{Col}(A) \) is

\[
\begin{bmatrix}
2 \\
-1 \\
-4 \\
3
\end{bmatrix}
\]

and using

\[
\begin{bmatrix}
2 & -6 & 0 \\
-1 & 3 & 0 \\
-4 & 12 & 0 \\
3 & -9 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & -3 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

we see that a non–zero vector in \( \text{Nul}(A) \) is

\[
\begin{bmatrix}
3 \\
1
\end{bmatrix}
\]

23. To determine if \( w \) is in \( \text{Col}(A) \), we must determine whether or not the equation \( Ax = w \) has a solution. Using

\[
\begin{bmatrix}
-6 & 12 & 2 \\
-3 & 6 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & -2 & -\frac{1}{3} \\
0 & 0 & 0
\end{bmatrix}
\]

we conclude that \( w \in \text{Col}(A) \). In fact, we see that
\[
\mathbf{w} = A \begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix}.
\]

(Thus \( \mathbf{w} \) is \(-1/3\) times the vector in the first column of \( A \).)

Since
\[
A \mathbf{w} = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]
we see that \( \mathbf{w} \) is also in \( \text{Nul}(A) \).

25.
\begin{enumerate}
\item a. True.
\item b. False. It is a subspace of \( \mathbb{R}^n \).
\item c. True.
\item d. False (or at least unclear) as stated. A true statement is: If the equation \( A\mathbf{x} = \mathbf{b} \) is consistent for every vector \( \mathbf{b} \in \mathbb{R}^m \), then \( \text{Col}(A) \) is \( \mathbb{R}^m \).
\item e. True.
\item f. True.
\end{enumerate}

26.
\begin{enumerate}
\item a. True.
\item b. True.
\item c. False. \( \text{Col}(A) \) is the set of all vectors \( \mathbf{b} \in \mathbb{R}^m \) for which the equation \( A\mathbf{x} = \mathbf{b} \) has a solution.
\item d. True.
\item e. True.
\item f. True.
\end{enumerate}

27. The given solution, \( \mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \), is in the null space of the coefficient matrix of the given system. Since this null space is a vector space, it must contain the vector \( 10\mathbf{x} \).

28. The fact that the first system has a solution means that the vector
\[
\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}
\]
is in the column space of the coefficient matrix of the system.

Since the column space is a vector space, it must contain the vector \( 5\mathbf{b} \).

31.
\begin{enumerate}
\item a. For any \( p \) and \( q \in P_2 \), we have
\[ T(p + q) = \begin{bmatrix} (p + q)(0) \\ (p + q)(1) \end{bmatrix} = \begin{bmatrix} p(0) + q(0) \\ p(1) + q(1) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(0) \\ q(1) \end{bmatrix} = T(p) + T(q). \]

Also, for any \( p \in P_2 \) and for any scalar \( c \), we have
\[
T(cp) = \begin{bmatrix} (cp)(0) \\ (cp)(1) \end{bmatrix} = \begin{bmatrix} c \cdot p(0) \\ c \cdot p(1) \end{bmatrix} = c \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = cT(p).
\]

This shows that \( T \) is a linear transformation.

b. The kernel of \( T \) consists of all functions \( p \in P_2 \) such that \( p(0) = 0 \) and \( p(1) = 0 \). Since every \( p \in P_2 \) has the form \( p(t) = at^2 + bt + c \), we see that \( p \in \ker(T) \) if and only if both
\[ a(0)^2 + b(0) + c = 0 \]
and
\[ a(1)^2 + b(1) + c = 0. \]
This is true if any only if \( c = 0 \) and \( a = -b \). Thus, \( \ker(T) \) consists of all \( p \in P_2 \) that have the form
\[ p(t) = at^2 - at \]
or equivalently
\[ p(t) = a(t^2 - t) \]
(\( a \) can be any scalar). In conclusion,
\[ \ker(T) = \text{Span}(f) \]
where \( f \) is the function \( f(t) = t^2 - t \).
Since every \( p \in P_2 \) has the form \( p(t) = at^2 + bt + c \), the range of \( T \) consists of all vectors in \( \mathbb{R}^2 \) that have the form
\[
\begin{bmatrix}
  a(0)^2 + b(0) + c \\
  a(1)^2 + b(1) + c
\end{bmatrix} = \begin{bmatrix}
  c \\
  a + b + c
\end{bmatrix}
\]
\[
= a \begin{bmatrix}
  0 \\
  1
\end{bmatrix} + b \begin{bmatrix}
  0 \\
  1
\end{bmatrix} + c \begin{bmatrix}
  1 \\
  1
\end{bmatrix}
\]
\[
= d \begin{bmatrix}
  0 \\
  1
\end{bmatrix} + c \begin{bmatrix}
  1 \\
  1
\end{bmatrix}
\]
(where \(d\) and \(c\) can be any scalars). Thus,
\[
\text{Range}(T) = \text{Span}\left\{ \begin{bmatrix}
  0 \\
  1
\end{bmatrix}, \begin{bmatrix}
  1 \\
  1
\end{bmatrix} \right\} = \mathbb{R}^2.
\]
(Note that Range\((T)\) is two–dimensional and \(\text{ker}(T)\) is one–dimensional – which makes sense because \(P_2\) is three–dimensional.)

33.

a. For any \(A\) and \(B \in M_{2 \times 2}\), we have
\[
T(A + B) = (A + B) + (A + B)^T
\]
\[
= A + B + A^T + B^T
\]
\[
= (A + A^T) + (B + B^T)
\]
\[
= T(A) + T(B).
\]

Also, for any \(A \in M_{2 \times 2}\) and any scalar \(c\), we have
\[
T(cA) = cA + (cA)^T
\]
\[
= cA + c(A^T)
\]
\[
= c(A + A^T)
\]
\[
= cT(A).
\]

This shows that \(T\) is a linear transformation.

b. Supposing that \(B\) is a matrix in \(M_{2 \times 2}\) such that \(B^T = B\), we want to find a matrix \(A \in M_{2 \times 2}\) such that \(T(A) = B\). By doing some experimentation, we notice that
\[
T(B) = B + B^T = 2B.
\]
Since \(T(B) = 2B\) and \(T\) is a linear transformation, then
\[
T\left( \frac{1}{2} B \right) = \frac{1}{2} T(B) = \frac{1}{2} (2B) = B.
\]
Thus, for \(A = \frac{1}{2} B\), we have \(T(A) = B\).

c. If \(B \in \text{Range}(T)\), then then there exists some matrix \(A \in M_{2 \times 2}\) such that \(B = A + A^T\). This means that
\[
B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = B.
\]
Conversely, if $B^T = B$, then from what was shown in part b, we know that $T\left(\frac{1}{2}B\right) = B$ which means that $B \in \text{Range}(T)$.

In conclusion, Range($T$) consists of all matrices $B \in M_{2 \times 2}$ such that $B^T = B$.

d. ker($T$) consists of all matrices $A \in M_{2 \times 2}$ such that $A + A^T = 0$ (the $2 \times 2$ zero matrix). In general, a matrix $A \in M_{2 \times 2}$ looks like

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

In order to have $A + A^T = 0$, we must have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$ 

This will be true if and only if $a = 0$, $d = 0$, and $b + c = 0$. We conclude that ker($T$) consists of all matrices of the form

$$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

(where $b$ can be any scalar). Another way to state this is that

$$\text{ker}(T) = \text{Span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$ 

We remark that ker($T$) is one–dimensional. Since $M_{2 \times 2}$ is four–dimensional, it should be the case that Range($T$) is three–dimensional. Recall that Range($T$) consists of all $B \in M_{2 \times 2}$ such that $B^T = B$. Letting

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we see that $B^T = B$ if and only if

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which is true if and only if $b = c$. Range($T$) thus consists of all matrices of the form

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which shows that

$$\text{Range}(T) = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$
Since these three matrices are also linearly independent, we see that \( \text{Range}(T) \) is three-dimensional as expected.