Solutions to Selected Section 4.4 Homework Problems
Problems 1-21 (odd) and 16.

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1. Recall that if $B = \{b_1, b_2, \ldots, b_n\}$ is a basis for $\mathbb{R}^n$ and $P_B$ is the matrix
   \[ P_B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}, \]
   then for any vector $x \in \mathbb{R}^n$, we have
   \[ x = P_B [x]_B \]
   and
   \[ [x]_B = P_B^{-1} x. \]

For the basis and coordinate vector given in this problem, we have
\[ x = P_B [x]_B \]
\[ = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \]
\[ = \begin{bmatrix} 3 \\ -7 \end{bmatrix}. \]

5. \[ [x]_B = P_B^{-1} x \]
\[ = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \]
\[ = \begin{bmatrix} -5 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \]
\[ = \begin{bmatrix} 8 \\ -5 \end{bmatrix}. \]

9. The change of coordinates matrix from the basis, $B$, given in this problem to the standard basis is
   \[ P_B = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}. \]
\[ [x]_B = P_B^{-1}x \]
\[ = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -6 \end{bmatrix} \]
\[ = \begin{bmatrix} -3 & -2 \\ -\frac{5}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -6 \end{bmatrix} \]
\[ = \begin{bmatrix} 6 \\ 4 \end{bmatrix}. \]

We can check that this correct by observing that
\[ \begin{bmatrix} 2 \\ -6 \end{bmatrix} = 6 \begin{bmatrix} 3 \\ -5 \end{bmatrix} + 4 \begin{bmatrix} -4 \\ 6 \end{bmatrix}. \]

13. We want to find \( c_1, c_2, \) and \( c_3 \) such that
\[ c_1(1 + t^2) + c_2(t + t^2) + c_3(1 + 2t + t^2) = 1 + 4t + 7t^2. \]
If we write this equation as
\[ (c_1 + c_3) + (c_2 + 2c_3)t + (c_1 + c_2 + c_3)t^2 = 1 + 4t + 7t^2, \]
we see that we need to solve the linear system
\[ c_1 + c_3 = 1 \]
\[ c_2 + 2c_3 = 4 \]
\[ c_1 + c_2 + c_3 = 7. \]

Since
\[ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \]
we see that \( c_1 = 2, c_2 = 6, \) and \( c_3 = -1. \) Thus, the coordinate vector of the function \( p \) relative to the basis \( B \) is
\[ [p]_B = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}. \]

As a check, observe that
\[ 2(1 + t^2) + 6(t + t^2) - (1 + 2t + t^2) = 1 + 4t + 7t^2. \]

15.

a. True.

b. False. See what I wrote in the solution to problem 1.

c. False. The vector spaces \( P_2 \) and \( \mathbb{R}^3 \) are isomorphic, however.
16. 
  a. True.
  b. False. The correspondence $x \mapsto [x]_B$ is called the coordinate mapping. The correspondence $[x]_B \mapsto x$ is the inverse of the coordinate mapping.
  c. True. In fact, all planes in $\mathbb{R}^3$ are isomorphic to $\mathbb{R}^2$ in the sense that there always exists a one–to–one mapping of a plane in $\mathbb{R}^3$ onto $\mathbb{R}^2$. However, our definition of “isomorphism” pertains only to vector spaces. A plane in $\mathbb{R}^3$ is a vector space if and only if this plane passes through the origin in $\mathbb{R}^3$. According to our definition, only planes in $\mathbb{R}^3$ that pass through the origin in $\mathbb{R}^3$ are isomorphic to $\mathbb{R}^2$. In other branches of mathematics (notably Topology), an isomorphism between two sets $A$ and $B$ is simply defined to be a one–to–one mapping of $A$ onto $B$. According to this definition, all planes in $\mathbb{R}^3$ are isomorphic to $\mathbb{R}^2$.

17. Note that

\[
\begin{bmatrix}
1 & 2 & -3 & 1 \\
-3 & -8 & 7 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -5 & 5 \\
0 & 1 & 1 & -2
\end{bmatrix}.
\]

Thus, we see, for example that

\[
5\begin{bmatrix}
1 \\
-3
\end{bmatrix} - 2\begin{bmatrix}
2 \\
-8
\end{bmatrix} + 0\begin{bmatrix}
-3 \\
7
\end{bmatrix} = \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

and

\[
10\begin{bmatrix}
1 \\
-3
\end{bmatrix} - 3\begin{bmatrix}
2 \\
-8
\end{bmatrix} + 1\begin{bmatrix}
-3 \\
7
\end{bmatrix} = \begin{bmatrix}
1 \\
1
\end{bmatrix}.
\]

19. Suppose that $V$ is a vector space and suppose that $S = \{b_1, b_2, \ldots, b_n\}$ is a set of vectors in $V$ with the property that every vector in $V$ can be expressed uniquely as a linear combination of the vectors in $S$.

We want to prove that $S$ is a basis for $V$. This means that there are two things to prove: that $S$ is a linearly independent set and that $S$ spans $V$.

To prove that $S$ is linearly independent, we consider the equation

\[c_1b_1 + c_2b_2 + \cdots + c_nb_n = 0.\]

We know that one solution of the above equation is the trivial solution: $c_1 = c_2 = \cdots = c_n = 0$. However, we are hypothesizing that every vector in $V$ (including the zero vector) can be expressed in one and only one way as a linear combination of the vectors in $S$. Thus, the trivial solution must be the only solution of the above equation. This tells us that $S$ is linearly independent.

The fact that $S$ spans $V$ follows immediately from our hypothesis, which says that every vector in $V$ can be expressed as a linear combination of the vectors in $S$. 

\[3\]
We conclude that $S$ is a basis for $V$.

21. Since each vector $x \in \mathbb{R}^2$ satisfies

$$[x]_B = P_B^{-1}x,$$

the matrix of the linear transformation $x \mapsto [x]_B$ is

$$P_B^{-1} = \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}.$$