The Row Space of a Matrix

Definition The row space of an $m \times n$ matrix, $A$, denoted by $\text{row}(A)$ is the set of all linear combinations of the row vectors of $A$.

Definition The column space of an $m \times n$ matrix, $A$, denoted by $\text{col}(A)$ is the set of all linear combinations of the column vectors of $A$.

Definition The null space of an $m \times n$ matrix, $A$, denoted by $\text{nul}(A)$ is the set of all solutions, $x$, of the equation $Ax = 0_m$.

Here are some basic observations about the row space, column space, and null space:

1. If $A$ is an $m \times n$ matrix, then $\text{col}(A)$ is a subspace of $\mathbb{R}^m$ and $\text{row}(A)$ is a subspace of $\mathbb{R}^n$. In particular, $\text{col}(A)$ is the span of the columns of $A$ and $\text{row}(A)$ is the span of the rows of $A$.

2. If $A$ is an $m \times n$ matrix, then $\dim(\text{col}(A)) + \dim(\text{nul}(A)) = n$.

3. If $A$ is an $m \times n$ matrix, then $\text{row}(A) = \text{col}(A^T)$ and $\text{col}(A) = \text{row}(A^T)$.
Example  For the $4 \times 5$ matrix

\[
A = \begin{bmatrix}
-2 & -5 & 8 & 0 & -17 \\
1 & 3 & -5 & 1 & 5 \\
3 & 11 & -19 & 7 & 1 \\
1 & 7 & -13 & 5 & -3
\end{bmatrix},
\]

find bases for $\text{col}(A)$, $\text{nul}(A)$, and $\text{row}(A)$.

Solution  From earlier work that we have done, we know how to find bases for $\text{col}(A)$ and $\text{nul}(A)$.

Since

\[
A = \begin{bmatrix}
-2 & -5 & 8 & 0 & -17 \\
1 & 3 & -5 & 1 & 5 \\
3 & 11 & -19 & 7 & 1 \\
1 & 7 & -13 & 5 & -3
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 1 & -2 & 0 & 3 \\
0 & 0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

we see that the first, second, and fourth columns of $A$ are the pivot columns of $A$. We also know that the pivot columns of $A$ form a basis for $\text{col}(A)$. Thus, $\text{col}(A)$ is three–dimensional and a basis for $\text{col}(A)$ is

\[
\left\{ \begin{bmatrix}
-2 \\
1 \\
3 \\
1
\end{bmatrix}, \begin{bmatrix}
-5 \\
3 \\
11 \\
7
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
7 \\
5
\end{bmatrix} \right\}.
\]

To find a basis for $\text{nul}(A)$, we observe that every solution of $Ax = 0_4$ has the form

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-s - t \\
-3s + 2t \\
t \\
5s \\
s
\end{bmatrix} = t \begin{bmatrix}
-1 \\
2 \\
1 \\
0 \\
0
\end{bmatrix} + s \begin{bmatrix}
-1 \\
-3 \\
0 \\
5 \\
1
\end{bmatrix}.
\]

Therefore, $\text{nul}(A)$ is two–dimensional and a basis for $\text{nul}(A)$ is

\[
\left\{ \begin{bmatrix}
-1 \\
2 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-1 \\
-3 \\
0 \\
5 \\
1
\end{bmatrix} \right\}.
\]

To find a basis for $\text{row}(A)$, we use the fact that $\text{row}(A) = \text{col}(A^T)$. Observing that
we see that the first, second, and fourth columns of $A^T$ are the pivot columns of $A^T$. Therefore, $\text{col}(A^T) = \text{row}(A)$ is three–dimensional and a basis for $\text{col}(A^T) = \text{row}(A)$ is

$$\left\{ \begin{bmatrix} -2 \\ -5 \\ 8 \\ 0 \\ -17 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -13 \\ 5 \\ -3 \end{bmatrix} \right\}. $$
How to Find a Basis for $\text{col}(A)$ and $\text{row}(A)$ in One Fell Swoop

If $A$ is an $m \times n$ matrix, then every vector $\mathbf{v} \in \text{col}(A)$ is a linear combination of the columns of $A$. This means that $\mathbf{v} = A\mathbf{x}$ for some vector $\mathbf{x} \in \mathbb{R}^n$. Likewise, every vector $\mathbf{w} \in \text{row}(A)$ is a linear combination of the rows of $A$. This means that $\mathbf{w} = A^\top \mathbf{y}$ for some vector $\mathbf{y} \in \mathbb{R}^m$. We will use this way of looking at things to show that the row spaces of any two equivalent matrices are the same!

**Lemma**  If $A$ and $B$ are $m \times n$ matrices and $A \sim B$, then there exists an invertible $m \times m$ matrix, $C$, such that $B = CA$. In particular, if $A$ is any $m \times n$ matrix, then there exists an invertible $m \times m$ matrix $C$ such that $\text{rref}(A) = CA$.

**Proof**  The fact that $A \sim B$ means that if we start with $A$ and perform a certain sequence of elementary row operations, then we arrive at $B$. This means that there is a sequence, $E_1, E_2, \ldots, E_k$ of elementary $m \times m$ matrices such that

$$E_k \cdots E_2 E_1 A = B.$$  

However, recall that all elementary matrices are invertible and recall that any product of invertible matrices is invertible. This means that the matrix

$$C = E_k \cdots E_2 E_1$$

is invertible.

We have shown that $B = CA$ where $C$ is an invertible $m \times m$ matrix.

**Example**  As an example to illustrate the above lemma, let us consider the process of finding $\text{rref}(A)$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$  

We do this, step by step, as follows:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \text{rref}(A).$$
In terms of elementary matrices, this procedure is written as
\[
\begin{bmatrix}
1 & -2 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -\frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
4 & 5 & 6
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 2
\end{bmatrix},
\]
which can be written as
\[
\begin{bmatrix}
-\frac{5}{3} & \frac{2}{3} \\
\frac{4}{3} & -\frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 2
\end{bmatrix}.
\]
Thus, in this example we see that \( \text{rref}(A) = CA \) where \( C \) is the matrix
\[
C =
\begin{bmatrix}
-\frac{5}{3} & \frac{2}{3} \\
\frac{4}{3} & -\frac{1}{3}
\end{bmatrix}.
\]
**Theorem**  If $A$ and $B$ are $m \times n$ matrices and $A \sim B$, then $\text{row}(A) = \text{row}(B)$. In particular, if $A$ is any $m \times n$ matrix, then $\text{row}(A) = \text{row}(\text{rref}(A))$.

**Proof**  Since $A \sim B$, then we know that there exists an invertible $m \times m$ matrix $C$ such that $B = CA$.

We will now show that $\text{row}(B) \subseteq \text{row}(A)$.

Suppose that $w \in \text{row}(B)$. This means that there exists a vector $y \in \mathbb{R}^m$ such that $w = B^T y$. From this, we obtain

$$w = (CA)^T y$$

$$\Rightarrow w = (A^T C^T) y$$

$$\Rightarrow w = A^T (C^T y).$$

Since the vector $C^T y$ is in $\mathbb{R}^m$, the last equation above shows us that $w$ is a linear combination of the columns of $A^T$. This means that $w \in \text{row}(A)$. We have proved that $\text{row}(B) \subseteq \text{row}(A)$.

We will now show that $\text{row}(A) \subseteq \text{row}(B)$.

Suppose that $w \in \text{row}(A)$. This means that there exists a vector $y \in \mathbb{R}^m$ such that $w = A^T y$. From this, we obtain

$$w = (C^{-1} B)^T y$$

$$\Rightarrow w = (B^T (C^{-1})^T) y$$

$$\Rightarrow w = B^T (C^{-1})^T y).$$

Since the vector $(C^{-1})^T y$ is in $\mathbb{R}^m$, the last equation above shows us that $w$ is a linear combination of the columns of $B^T$. This means that $w \in \text{row}(B)$. We have proved that $\text{row}(A) \subseteq \text{row}(B)$.

Since $\text{row}(B) \subseteq \text{row}(A)$ and $\text{row}(A) \subseteq \text{row}(B)$, then $\text{row}(A) = \text{row}(B)$.

**Theorem**  If $A$ is an $m \times n$ matrix, then the non–zero rows of $\text{rref}(A)$ form a basis for $\text{row}(A)$ (and also form a basis for $\text{row}(\text{rref}(A))$ since $\text{row}(A) = \text{row}(\text{rref}(A))$).
Example  Use the above theorem to find a basis for the row space of the matrix

\[
A = \begin{bmatrix}
-2 & -5 & 8 & 0 & -17 \\
1 & 3 & -5 & 1 & 5 \\
3 & 11 & -19 & 7 & 1 \\
1 & 7 & -13 & 5 & -3
\end{bmatrix}.
\]

Solution  Since

\[
\text{rref}(A) = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 1 & -2 & 0 & 3 \\
0 & 0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

we see that a basis for row(A) is

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ -5 \end{bmatrix} \right\}.
\]

(Note that this basis for row(A) is different than the one that we found when we did this problem earlier using a different method.)
Example  Find bases for the column space, null space, and row space of the matrix

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}.
\]

Solution  Since

\[
rref(A) = \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 2
\end{bmatrix},
\]

we see that a basis for \(\text{col}(A)\) is

\[
\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\},
\]

a basis for \(\text{nul}(A)\) is

\[
\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\},
\]

and a basis for \(\text{row}(A)\) is

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}.
\]
**Remark** If $A$ and $B$ are $m \times n$ matrices and $A \sim B$, then it is **not** necessarily true that $\text{col}(A) = \text{col}(B)$. In particular, it is not necessarily true that $\text{col}(A) = \text{col}(\text{rref}(A))$. However, it is always true that $\dim(\text{col}(A)) = \dim(\text{col}(\text{rref}(A)))$.

**Theorem** If $A$ is any $m \times n$ matrix, then the row space and the column space of $A$ have the same dimension.

**Proof** The dimension of the column space of $A$ is the number of pivot columns of $\text{rref}(A)$. The dimension of the row space of $A$ is the number of non–zero rows of $\text{rref}(A)$. However, the number of pivot columns of $\text{rref}(A)$ is the same as the number of non–zero rows of $\text{rref}(A)$. Therefore, $\dim(\text{col}(A)) = \dim(\text{row}(A))$.

**Example** For the matrix

$$
A = \begin{bmatrix}
-2 & -5 & 8 & 0 & -17 \\
1 & 3 & -5 & 1 & 5 \\
3 & 11 & -19 & 7 & 1 \\
1 & 7 & -13 & 5 & -3 \\
\end{bmatrix},
$$
we have seen that $\dim(\text{col}(A)) = \dim(\text{row}(A)) = 3$.

For the matrix

$$
A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{bmatrix},
$$
we have seen that $\dim(\text{col}(A)) = \dim(\text{row}(A)) = 2$.

**Definition** If $A$ is an $m \times n$ matrix, then we define the **rank** of $A$, denoted by $\text{rank}(A)$, to be the dimension of the column space of $A$ (which is the same as the dimension of the row space of $A$). If $A$ is a square matrix of size $n \times n$ and $\text{rank}(A) = n$, then we say that $A$ has **full rank**.

**Theorem** An $n \times n$ matrix, $A$, is invertible if and only if $A$ has full rank.

**Remark** Any of the many other statements (for example, $A \sim I_n$) that are given in the “Square Matrix Theorem” are equivalent to the statement that $A$ has full rank.