1. (Definitions) Use complete sentences to write the following definitions.

(a) What is a linear equation in the variables $x_1, x_2, \ldots, x_n$?

(b) What is meant by an elementary row operation performed on a matrix?

(c) What is a pivot position in a matrix? What is a pivot column in a matrix?

(d) What is meant by a linear combination of a set of vectors, $u_1, u_2, \ldots, u_n$ in $V_m$?

(e) What is a matrix equation?
2. Use step-by-step hand calculation (doing one step at a time and explaining each step as you go along using the notation that we have been using) to find the reduced echelon form of the matrix

\[
A = \begin{bmatrix}
-1 & -2 & 0 \\
1 & -1 & 1 \\
-2 & 1 & -1 \\
2 & -2 & 1
\end{bmatrix}.
\]

The answer that you should obtain is

\[
\text{rref}(A) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.
\]

You should be able to obtain this result using a sequence of eleven elementary row operations.
(a) Give an example of an inconsistent system of linear equations with four equations and two unknowns or, if it is not possible to give an example of such a system, then explain why it is not possible.

(b) Give an example of a system of linear equations with four equations and two unknowns that has a unique solution or, if it is not possible to give an example of such a system, then explain why it is not possible.

(c) Give an example of a system of linear equations with four equations and two unknowns that has infinitely many solutions or, if it is not possible to give an example of such a system, then explain why it is not possible.
Let \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) be the vectors

\[
\mathbf{u}_1 = \begin{bmatrix} -5 \\ 7 \\ 6 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -6 \\ 0 \\ 5 \end{bmatrix}.
\]

For each of the vectors, \( \mathbf{v} \), \( \mathbf{w} \), and \( \mathbf{z} \) given in parts a, b, and c below, decide whether or not the vector is in \( \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\} \). If the vector is in \( \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\} \), then show how to write the vector as a linear combination of \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \). (That is, find weights, \( c_1 \) and \( c_2 \), such that the given vector is equal to \( c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 \).) If the given vector is not in \( \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\} \), then explain why not.

(a) \[
\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

(b) \[
\mathbf{w} = \begin{bmatrix} -10 \\ 1 \\ 1 \end{bmatrix}
\]

(c) \[
\mathbf{z} = \begin{bmatrix} -3 \\ -10 \\ 5 \end{bmatrix}
\]

(d) Is \( \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\} \) a line or a plane in \( \mathbb{R}^3 \)? Explain your answer.
Decide whether or not the columns of the matrix

\[
A = \begin{bmatrix}
1 & 3 & -2 & 2 \\
0 & 1 & 1 & -5 \\
1 & 2 & -3 & 7 \\
-2 & -8 & 2 & -1
\end{bmatrix}
\]

span \( \mathbb{R}^4 \).
3. Write a system of linear equations that describes the traffic network pictured below.

Find the general solution of this system and then find the particular solution that corresponds to a flow of $x_1 = 20$ cars per minute.
4. Decide whether each of the following statements is true or false.

(a) It is possible for a system of three linear equations in three unknowns to have infinitely many solutions. (True, False)

(b) It is possible for a system of three linear equations in three unknowns to have no solutions. (True, False)

(c) The vector \[
\begin{bmatrix}
12 \\
-3
\end{bmatrix}
\] is a linear combination of the vectors \[
\begin{bmatrix}
4 \\
-1
\end{bmatrix}
\text{ and }
\begin{bmatrix}
-4 \\
-1
\end{bmatrix}.
\] (True, False)

(d) If every column of the augmented matrix \[A \ b\] is a pivot column, then the equation \[A x = b\] is consistent. (True, False)

(e) It is possible that two matrices, \(A\) and \(B\), are row–equivalent to each other, and yet \(A\) and \(B\) still might not have the same reduced echelon form. (True, False)

(f) Some homogeneous systems of linear equations are inconsistent. (True, False)

(g) Given any three vectors \(u_1, u_2,\) and \(u_3 \in V_n\), it must be true that \[u_3 \in \text{Span} \{u_1, u_2, u_3\}.\] (True, False)

(h) If the equations \(A x = b\) and \(A x = c\) are both consistent and the equation \(A x = b\) has infinitely many solutions, then the equation \(A x = c\) also has infinitely many solutions. (True, False)

(i) If an augmented matrix \([A \ b]\) can be transformed into an augmented matrix \([C \ d]\) via a sequence of elementary row operations, then the equations \(A x = b\) and \(C x = d\) have exactly the same solution sets. (True, False)

(j) If a \(4 \times 4\) matrix has exactly four pivot positions, then \(\text{rref}(A) = I_4\) (the \(4 \times 4\) identity matrix). (True, False)