1. (Definitions) Use complete sentences to write the following definitions.

(a) Let \( \{v_1, v_2, \ldots, v_n\} \) be a set of vectors in \( V_m \). What do we mean when we say that this set of vectors is \textit{linearly dependent}?

(b) What is meant by a \textit{transformation} (or \textit{function} or \textit{mapping}), \( T \), from \( V_n \) to \( V_m \)? What are the \textit{domain} and \textit{codomain} of this transformation?

(c) What do we mean by the \textit{zero transformation} \( T : V_n \rightarrow V_m \)?

(d) What do we mean when we say that a transformation \( T : V_n \rightarrow V_m \) is \textit{one-to-one}?

(e) What do we mean when we say that a transformation \( T : V_n \rightarrow V_m \) is \textit{onto} \( V_m \)?
2. For the statement given below, decide if the statement is always true or if the statement might be false in some cases. If the statement is always true, then explain why. If the statement might be false in some cases, then give an example of a case in which the statement is false. (In other words, give a counterexample.) The statement is:

If \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \) and \( \mathbf{v}_4 \) are vectors in \( V_4 \) and \( \mathbf{v}_3 = \mathbf{0}_4 \), then the set of vectors \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \) is linearly dependent.
3. Let $T : V_2 \rightarrow V_2$ be the linear transformation defined by

$$T(x) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

for all $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in V_2$.

(a) Plot the vectors $u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

(b) Compute $T(u)$ and $T(v)$ and plot the vectors $T(u)$ and $T(v)$.

(c) Describe geometrically (writing in complete sentences) what the transformation $T$ does to each vector in $V_2$. \textit{(Note: Don’t just describe what $T$ does to the vectors $u$ and $v$ given above. Describe the general effect of $T$ on all vectors in $V_2$.)}
4. Let \( T : V_4 \rightarrow V_1 \) be the linear transformation defined by

\[
T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = [2x_1 + 3x_3 - 4x_4].
\]

(a) Find the standard matrix of this linear transformation.
(b) Is \( T \) onto \( V_1 \)? Explain why or why not. (No explanation = no credit.)
(c) Is \( T \) one–to–one? Explain why or why not. (No explanation = no credit.)
5. This year, Club A has 6000 members and Club B has 4000 members. Each year, 5% of Club A’s members quit Club A and join Club B. Also, each year, 5% of Club B’s members quit Club B and join Club A and 5% of Club B’s members quit club B and don’t join any other club. Set up a difference equation that describes this situation. (Make sure to state what all variables you are using stand for.) Then, by calculator computation, determine the number of members that each club will have ten years from now. (You do not have to write down every intermediate computation. But do show at least the first couple of iterations. Round your answers to the nearest whole number.)
6. Decide whether each of the following statements is true or false.

(a) The columns of a matrix \( A \) are linearly independent if the equation \( Ax = 0 \) has only the trivial solution. (True, False)

(b) A set of two vectors, \( \{v_1, v_2\} \), in \( V_4 \) is linearly dependent if and only if one of the two vectors is a scalar multiple of the other one. (True, False)

(c) If \( A \) is a \( 3 \times 5 \) matrix and \( T \) is a linear transformation defined by \( T(x) = Ax \), then the domain of \( T \) is \( V_3 \). (True, False)

(d) A transformation, \( T \), is a linear transformation if and only if

\[
T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)
\]

for all \( v_1 \) and \( v_2 \) in the domain of \( T \) and for all scalars \( c_1 \) and \( c_2 \). (True, False)

(e) If \( T : V_3 \to V_4 \) is a linear transformation, then the range of \( T \) must be \( V_4 \). (True, False)

(f) If \( T : V_3 \to V_3 \) is a linear transformation and \( x \) is any vector in \( V_3 \), then it must be true that \( T(x) \) is a scalar multiple of \( x \). (True, False)

(g) When two linear transformations are performed one after another, then the combined effect may not always be a linear transformation. (True, False)

(h) A linear transformation, \( T : V_n \to V_m \), is onto \( V_m \) if each vector in \( V_m \) is the image under \( T \) of at least one vector in \( V_n \). (True, False)

(i) The zero transformation, \( T : V_3 \to V_3 \) defined by \( T(x) = 0_3 \) for all \( x \in V_3 \) is not one-to-one. (True, False)

(j) The zero transformation, \( T : V_3 \to V_3 \) defined by \( T(x) = 0_3 \) for all \( x \in V_3 \) maps \( V_3 \) onto \( V_3 \). (True, False)