1. (Definitions) Use complete sentences to write the following definitions.

(a) Let

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

be a $2 \times 2$ matrix. What is meant by the determinant of $A$?

(b) What is the $n \times n$ identity matrix?

(c) What does it mean for an $n \times n$ matrix, $A$, to be invertible?

(d) What is meant by the inverse of an invertible $n \times n$ matrix, $A$?

(e) What is an elementary matrix?
(a) Suppose that $A$ is a $3 \times 4$ matrix whose columns span $V_3$ and suppose that $C$ is a $3 \times 3$ matrix. Explain how to construct a $4 \times 3$ matrix, $B$, such that $AB = C$. (This is an essay question. Write carefully and use complete sentences.)

(b) For the matrices

$$
A = \begin{bmatrix}
1 & 0 & 1 & 0 \\
1 & 2 & 0 & 1 \\
1 & 1 & 0 & 0 \\
\end{bmatrix}
\quad \text{and} \quad
C = \begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 2 \\
2 & -1 & 1 \\
\end{bmatrix},
$$

use the procedure that you described in part a to find a $4 \times 3$ matrix, $B$, such that $AB = C$. 

2. Let

\[
A = \begin{bmatrix}
4 & -10 & 1 & 2 \\
-10 & 4 & 4 & -2 \\
0 & 10 & -1 & -5 \\
-9 & 3 & 0 & 8 \\
\end{bmatrix}
\]

Write down the $4 \times 4$ elementary matrix, $E$, such that

\[
EA = \begin{bmatrix}
4 & -10 & 1 & 2 \\
-10 & 4 & 4 & -2 \\
0 & 10 & -1 & -5 \\
-27 & 9 & 0 & 24 \\
\end{bmatrix}.
\]

This should not require any computation. Just write down the matrix $E$ based on your knowledge of the effects of multiplying matrices on the left by elementary matrices.
3. Use the algorithm

\[
[A \quad I] \sim \cdots \sim [I \quad A^{-1}]
\]

to find the inverse of the matrix

\[
A = \begin{bmatrix}
1 & -2 & 1 \\
4 & -7 & 3 \\
-2 & 6 & -4
\end{bmatrix}
\]

if it exists. If \(A^{-1}\) does not exist, then explain how the algorithm tells you this. You may not use determinants or your calculator in doing this problem. You must use the algorithm!
4. Compute the determinant

\[
\begin{vmatrix}
3 & 1 & 3 \\
5 & -1 & -3 \\
2 & 3 & 0
\end{vmatrix}
\]

by performing a cofactor expansion along the first row. Then compute the determinant by performing a cofactor expansion along the first column. Show all of your computations.
5. For the matrices

\[
A = \begin{bmatrix}
-1 & -3 \\
4 & 5
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
1 & 1 \\
4 & -1
\end{bmatrix},
\]

verify by direct computation (showing all computations) that \( \det(AB) = \det(A) \det(B) \).
6. Use Cramer’s Rule to find the solution of the system

\[
\begin{align*}
3x_1 - 2x_2 &= 7 \\
-5x_1 + 6x_2 &= -5.
\end{align*}
\]

(You must use Cramer’s Rule – not some other method.)
7. Decide whether each of the following statements is true or false.

(a) If \( A \) is any \( n \times n \) matrix, then \( I_n A = A \). (True, False)

(b) If \( A, B, \) and \( C \) are any three \( n \times n \) matrices, then \( A (B + C) = AB + AC \). (True, False)

(c) The matrix
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}
\]
is an elementary matrix. (True, False)

(d) If \( A \) is a \( 3 \times 3 \) matrix and the equation
\[
Ax = \begin{bmatrix}
-2 \\
-3 \\
4
\end{bmatrix}
\]
has a infinitely many solutions, then \( A \) must be invertible. (True, False)

(e) If \( A \) is a square matrix that has a left inverse, then \( A \) must also have a right inverse and these left and right inverses must be equal to each other. (True, False)

(f) If two rows of a \( 5 \times 5 \) matrix, \( A \), are the same, then \( \det (A) = 0 \). (True, False)

(g) If \( A \) and \( B \) are square matrices of the same size and \( B \) is produced by interchanging two rows of \( A \), then \( \det (B) = -\det (A) \). (True, False)

(h) If \( A \) is any square matrix, then \( \det (A^T) = 1/\det (A) \). (True, False)

(i) If \( A \) is any square matrix, then \( \det (A) \det (A^{-1}) = 1 \). (True, False)

(j) Every elementary matrix has determinant 1 or \(-1\). (True, False)