Instructions. Remember to include all important details of your work. You will not get full credit (or perhaps even any partial credit) if I see gaps in your reasoning. Also, use correct notation and write in complete sentences where appropriate. You may use a calculator on this exam but you may not use any books or notes.

1. (Definitions) Use complete sentences to write the following definitions.

(a) Let

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

be a $2 \times 2$ matrix. What is meant by the **determinant** of $A$?

(b) What is the $n \times n$ **identity matrix**?

(c) What does it mean for an $n \times n$ matrix, $A$, to be **invertible**?

(d) What is meant by the **inverse** of an invertible $n \times n$ matrix, $A$?

(e) What is an **elementary matrix**?
(a) Suppose that $A$ is a $3 \times 4$ matrix whose columns span $V_3$ and suppose that $C$ is a $3 \times 3$ matrix. Explain how to construct a $4 \times 3$ matrix, $B$, such that $AB = C$. (This is an essay question. Write carefully and use complete sentences.)

(b) For the matrices

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & -1 & 1 \end{bmatrix},$$

use the procedure that you described in part a to find a $4 \times 3$ matrix, $B$, such that $AB = C$.

**Solution:** For the unknown matrix,

$$B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix},$$

it must be true that

$$AB = [Ab_1 \ Ab_2 \ Ab_3] = [c_1 \ c_2 \ c_3] = C.$$

Thus, to find $B$, we must solve all of the matrix equations $Ab_1 = c_1$, $Ab_2 = c_2$, and $Ab_3 = c_3$. (Solutions to all of these equations exist because the columns of $A$ span $V_3$.) Since all of these matrix equations have the same coefficient matrix

$$A = [a_1 \ a_2 \ a_3 \ a_4],$$

we can find $B$ in “one fell swoop” by row–reducing the “triple–augmented” matrix

$$[a_1 \ a_2 \ a_3 \ a_4 \ c_1 \ c_2 \ c_3].$$

Example: For the matrices, $A$ and $C$, given in part b above we have

$$[A \ C] = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & 4 & -2 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 & -3 & 4 & 0 \end{bmatrix}$$

and we see from this that one possible $B$ such that $AB = C$ is given by

$$B = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 1 \\ -3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This $B$ was obtained by assigning $x_4 = 0$ in each equation $Ab_i = c_i$. There are infinitely many other $B$ such that $AB = C$. For example, note that

$$B = \begin{bmatrix} 5 & -2 & -1 \\ -3 & 1 & 2 \\ -4 & 4 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

also works.
2. Let

\[ A = \begin{bmatrix} 4 & -10 & 1 & 2 \\ -10 & 4 & 4 & -2 \\ 0 & 10 & -1 & -5 \\ -9 & 3 & 0 & 8 \end{bmatrix} \]

Write down the $4 \times 4$ elementary matrix, $E$, such that

\[ EA = \begin{bmatrix} 4 & -10 & 1 & 2 \\ -10 & 4 & 4 & -2 \\ 0 & 10 & -1 & -5 \\ -27 & 9 & 0 & 24 \end{bmatrix}. \]

This should not require any computation. Just write down the matrix $E$ based on your knowledge of the effects of multiplying matrices on the left by elementary matrices.

**Answer:**

\[ E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}. \]
3. Use the algorithm

\[
[A \ I] \sim \cdots \sim [I \ A^{-1}]
\]

to find the inverse of the matrix

\[
A = \begin{bmatrix}
1 & -2 & 1 \\
4 & -7 & 3 \\
-2 & 6 & -4
\end{bmatrix}
\]

if it exists. If \(A^{-1}\) does not exist, then explain how the algorithm tells you this. You may not use determinants or your calculator in doing this problem. You must use the algorithm!

Solution:

\[
[A \ I] = \begin{bmatrix}
1 & -2 & 1 & 1 & 0 & 0 \\
4 & -7 & 3 & 0 & 1 & 0 \\
-2 & 6 & -4 & 0 & 0 & 1
\end{bmatrix}
\sim \begin{bmatrix}
1 & -2 & 1 & 1 & 0 & 0 \\
0 & 1 & -1 & -4 & 1 & 0 \\
-2 & 6 & -4 & 0 & 0 & 1
\end{bmatrix}
\sim \begin{bmatrix}
1 & -2 & 1 & 1 & 0 & 0 \\
0 & 1 & -1 & -4 & 1 & 0 \\
0 & 2 & -2 & 2 & 0 & 1
\end{bmatrix}
\sim \begin{bmatrix}
1 & -2 & 1 & 1 & 0 & 0 \\
0 & 1 & -1 & -4 & 1 & 0 \\
0 & 0 & 0 & 10 & -2 & 1
\end{bmatrix}
\]

shows that \(A\) is not row–equivalent to \(I\) and hence that \(A\) is not invertible.
4. Compute the determinant
\[
\begin{vmatrix}
3 & 1 & 3 \\
5 & -1 & -3 \\
2 & 3 & 0 \\
\end{vmatrix}
\]
by performing a cofactor expansion along the first row. Then compute the determinant by performing a cofactor expansion along the first column. Show all of your computations.

**Solution:** Expanding along the first row we obtain
\[
\begin{vmatrix}
3 & 1 & 3 \\
5 & -1 & -3 \\
2 & 3 & 0 \\
\end{vmatrix} = 3 \begin{vmatrix} -1 & -3 \\
3 & 0 \\
\end{vmatrix} - 1 \begin{vmatrix} 5 & -3 \\
2 & 0 \\
\end{vmatrix} + 3 \begin{vmatrix} 5 & -1 \\
2 & 3 \\
\end{vmatrix}
\]
\[
= 3 (9) - 1 (6) + 3 (17)
\]
\[
= 72.
\]

Expanding along the first column we obtain
\[
\begin{vmatrix}
3 & 1 & 3 \\
5 & -1 & -3 \\
2 & 3 & 0 \\
\end{vmatrix} = 3 \begin{vmatrix} -1 & -3 \\
3 & 0 \\
\end{vmatrix} - 5 \begin{vmatrix} 1 & 3 \\
3 & 0 \\
\end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\
-1 & -3 \\
\end{vmatrix}
\]
\[
= 3 (9) - 5 (-9) + 2 (0)
\]
\[
= 72.
\]
5. For the matrices

\[ A = \begin{bmatrix} -1 & -3 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}, \]

verify by direct computation (showing all computations) that \( \det(AB) = \det(A) \det(B) \).

**Solution:**

\[
\det(A) = (-1)(5) - (-3)(4) = 7
\]

and

\[
\det(B) = (1)(-1) - (1)(4) = -5.
\]

Also

\[
AB = \begin{bmatrix} -1 & -3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -13 & 2 \\ 24 & -1 \end{bmatrix}
\]

so

\[
\det(AB) = (-13)(-1) - (2)(24) = -35.
\]

This shows that \( \det(AB) = \det(A) \det(B) \).
6. Use Cramer’s Rule to find the solution of the system

\[ \begin{align*}
3x_1 - 2x_2 &= 7 \\
-5x_1 + 6x_2 &= -5.
\end{align*} \]

(You must use Cramer's Rule – not some other method.)

**Solution:** The coefficient matrix for this system is

\[ A = \begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}. \]

Note that \( \det(A) = 8 \). Also

\[ A_1(b) = \begin{bmatrix} 7 & -2 \\ -5 & 6 \end{bmatrix} \quad \text{and} \quad A_2(b) = \begin{bmatrix} 3 & 7 \\ -5 & -5 \end{bmatrix}. \]

By Cramer’s Rule, the solution of the given system is

\[ \begin{align*}
x_1 &= \frac{\det(A_1(b))}{\det(A)} = \frac{32}{8} = 4 \\
x_2 &= \frac{\det(A_2(b))}{\det(A)} = \frac{20}{8} = \frac{5}{2}.
\end{align*} \]
7. Decide whether each of the following statements is true or false.

(a) If $A$ is any $n \times n$ matrix, then $I_n A = A$.  \(\text{True}, \text{False}\)

(b) If $A$, $B$, and $C$ are any three $n \times n$ matrices, then $A (B + C) = AB + AC$.  \(\text{True}, \text{False}\)

(c) The matrix
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0 \\
\end{bmatrix}
\]
is an elementary matrix.  \(\text{True}, \text{False}\)

(d) If $A$ is a $3 \times 3$ matrix and the equation
\[
Ax = \begin{bmatrix}
-2 \\
-3 \\
4 \\
\end{bmatrix}
\]
has a infinitely many solutions, then $A$ must be invertible.  \(\text{True}, \text{False}\)

(e) If $A$ is a square matrix that has a left inverse, then $A$ must also have a right inverse and these left and right inverses must be equal to each other.  \(\text{True}, \text{False}\)

(f) If two rows of a $5 \times 5$ matrix, $A$, are the same, then $\det (A) = 0$.  \(\text{True}, \text{False}\)

(g) If $A$ and $B$ are square matrices of the same size and $B$ is produced by interchanging two rows of $A$, then $\det (B) = - \det (A)$.  \(\text{True}, \text{False}\)

(h) If $A$ is any square matrix, then $\det (A^T) = 1 / \det (A)$.  \(\text{True}, \text{False}\)

(i) If $A$ is any square matrix, then $\det (A) \det (A^{-1}) = 1$. \(\text{True}, \text{False}\)

(j) Every elementary matrix has determinant 1 or $-1$. \(\text{True}, \text{False}\)