1. (Definitions) Use complete sentences to write the following definitions.

(a) What is a **vector space**?

(b) What is a **subspace** of a vector space?

(c) Suppose that \( \{v_1, v_2, \ldots, v_n\} \) is a set of vectors in a vector space \( V \). What does it mean for this set of vectors to be **linearly independent**? What does it mean for this set of vectors to be **linearly dependent**?

(d) What is a **basis** for a vector space?

(e) Suppose that \( V \) is a vector space, that \( B = \{v_1, v_2, \ldots, v_n\} \) is a basis for \( V \), and \( v \) is a vector in \( V \). What is meant by the **coordinate vector** of \( v \) relative to the basis \( B \)?
2. Let \( P_2(R) \) be the set of all polynomial functions with domain \( R \) and with degree less than or equal to 2. (Thus a typical member of \( P_2(R) \) is a polynomial function of the form \( p(t) = at^2 + bt + c \) where \( a, b, \) and \( c \) can be any scalars.) We know that \( P_2(R) \) is a vector space.

Let \( H \) be the subset of \( P_2(R) \) consisting of all polynomial functions, \( p \), such that \( p(0) = 1 \). Is \( H \) a subspace of \( P_2(R) \) or not? If so, then explain why. If not, then explain why not. (No explanation=no credit.)

**Answer:** Recall that the zero vector of \( P_2(R) \) is the polynomial function

\[
z(t) = 0t^2 + 0t + 0.
\]

\( H \) is not a subspace of \( P_2(R) \) because \( z \notin H \). (Note that \( z(0) \neq 1 \).)
3. Let $A$ be the matrix

$$A = \begin{bmatrix} 4 & -1 & 9 \\ -1 & -1 & -1 \\ -2 & 3 & -7 \end{bmatrix}$$

and let $x$ be the vector

$$x = \begin{bmatrix} -6 \\ 0 \\ 3 \end{bmatrix}$$

Is $x$ in $Nul(A)$? Why or why not? (No explanation=no credit).

**Answer:** $x \notin Nul(A)$ because

$$Ax = \begin{bmatrix} 4 & -1 & 9 \\ -1 & -1 & -1 \\ -2 & 3 & -7 \end{bmatrix} \begin{bmatrix} -6 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -9 \end{bmatrix} \neq 0_3.$$
4. Suppose that $V$ and $W$ are vector spaces and suppose that $T : V \rightarrow W$ is a linear transformation. Also suppose that $\{v_1, v_2, \ldots, v_p\}$ is a linearly dependent set of vectors in $V$. Prove that the set of vectors $\{T(v_1), T(v_2), \ldots, T(v_p)\}$ is linearly dependent (in $W$).

**Proof:** Since the set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent in $V$, then there exists a dependence relation

$$c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0_V$$

(in which not all of the coefficients $c_1, c_2, \ldots, c_p$ are zero). Also, since $T$ is a linear transformation, then

$$c_1T(v_1) + c_2T(v_2) + \cdots + c_pT(v_p) = T(c_1v_1) + T(c_2v_2) + \cdots + T(c_pv_p)$$
$$= T(c_1v_1 + c_2v_2 + \cdots + c_pv_p)$$
$$= T(0_V)$$
$$= 0_W.$$  
(In the last equation above, we have used the fact that linear transformations always map zero vectors to zero vectors.) We have thus produced a dependence relation

$$c_1T(v_1) + c_2T(v_2) + \cdots + c_pT(v_p) = 0_W$$

(in which not all of the coefficients $c_1, c_2, \ldots, c_p$ are zero). This shows that the set of vectors $\{T(v_1), T(v_2), \ldots, T(v_p)\}$ is linearly dependent.
5. Explain why the set of vectors (functions) \( B = \{p_1, p_2, p_3\} \) where
\[
\begin{align*}
p_1(t) &= 2t^2 - 5t + 1 \\
p_2(t) &= 2t^2 + t \\
p_3(t) &= 2t^2 - 2t + 1
\end{align*}
\]
is a basis for \( P_2(R) \) and find the coordinate vector with respect to \( B \) of the function \( p \in P_2(R) \) defined by
\[
p(t) = -12t^2 + 12t - 3.
\]

Solution: The equation
\[
c_1p_1 + c_2p_2 + c_3p_3 = p
\]
is equivalent to
\[
c_1(2t^2 - 5t + 1) + c_2(2t^2 + t) + c_3(2t^2 - 2t + 1) = -12t^2 + 12t - 3 \quad \text{for all } t \in R
\]
and this is equivalent to
\[
(2c_1 + 2c_2 + 2c_3)t^2 + (-5c_1 + c_2 - 2c_3)t + (c_1 + c_3) = -12t^2 + 12t - 3 \quad \text{for all } t \in R.
\]
In order for this to hold, we must have
\[
\begin{align*}
2c_1 + 2c_2 + 2c_3 &= -12 \\
-5c_1 + c_2 - 2c_3 &= 12 \\
c_1 + c_3 &= -3.
\end{align*}
\]

Since
\[
\begin{bmatrix}
2 & 2 & 2 & -12 \\
-5 & 1 & -2 & 12 \\
1 & 0 & 1 & -3
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & -3 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]
we see that \( B \) is a basis for \( P_2(R) \) (because the non-augmented part of the above matrix is equivalent to \( I_3 \)) and we also see that
\[
-3p_1 - 3p_2 + 0p_3 = p
\]
meaning that
\[
[p]_B = \begin{bmatrix}
-3 \\
-3 \\
0
\end{bmatrix}.
\]
6. Decide whether each of the following statements is true or false.

(a) If $V$ is a vector space and $\mathbf{u}$ and $\mathbf{v}$ are any two vectors in $V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. (True, False)

(b) If $V$ is a vector space with zero vector $\mathbf{0}$ and $H$ is a subset of $V$ such that $\mathbf{0} \in H$, then $H$ must be a subspace of $V$. (True, False)

(c) If $V$ and $W$ are vector spaces and $T : V \rightarrow W$ is a linear transformation, then $\ker (T)$ is a subspace of $V$. (True, False)

(d) If $V$ and $W$ are vector spaces and $T : V \rightarrow W$ is linear transformation, then $\text{range } (T)$ is a subspace of $V$. (True, False)

(e) If $A$ is a $5 \times 4$ matrix, then $\text{Col} (A)$ is a subspace of $V_5$ (True, False)

(f) If $A$ is a $5 \times 4$ matrix, then $\text{Nul} (A)$ is a subspace of $V_4$ (True, False)

(g) If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are any three non–zero vectors in $V_3$, then $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $V_3$. (True, False)

(h) If $A$ is an $m \times n$ matrix, then the pivot columns of $A$ form a basis for $\text{Nul} (A)$. (True, False)

(i) If $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $V_3$ and $\mathbf{v} = \mathbf{v}_1 + 2\mathbf{v}_2 - 4\mathbf{v}_3$, then the coordinate vector of $\mathbf{v}$ with respect to the basis $B$ is

$[\mathbf{v}]_B = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$. (True, False)

(j) If $V$ is a vector space and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $V$, then the coordinate mapping $T : V \rightarrow V_3$ defined by $T (\mathbf{v}) = [\mathbf{v}]_B$ for all $\mathbf{v} \in V$ is an isomorphism (True, False).