Row Reduction and Echelon Forms

S. F. Ellermeyer

May 24, 2009

These notes closely follow the presentation of the material given in David C. Lay’s textbook *Linear Algebra and its Applications (3rd edition)*. These notes are intended primarily for in-class presentation and should not be regarded as a substitute for thoroughly reading the textbook itself and working through the exercises therein.

**Row Echelon Form and Reduced Row Echelon Form**

A non–zero row of a matrix is defined to be a row that does not contain all zeros.

The leading entry of a non–zero row of a matrix is defined to be the leftmost non–zero entry in the row.

For example, if we have the matrix

\[
\begin{bmatrix}
0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3
\end{bmatrix},
\]

then the first row is a non–zero row with leading entry 4, the second row is a zero row, and the third row is a non–zero row with leading entry 3.

**Definition 1** A matrix is said to have echelon form (or row echelon form) if it has the following properties:

1. All non–zero rows are above any zero rows.
2. Each leading entry of a non-zero row is in a column to the right of the leading entry of the row above it.

If a matrix has row echelon form and also satisfies the following two conditions, then the matrix is said to have **reduced echelon form** (or **reduced row echelon form**):

3. The leading entry in each non-zero row is 1.

4. Each leading 1 is the only non-zero entry in its column.

### 0.1 Quiz

Decide whether or not each of the following matrices has row echelon form. For each that does have row echelon form, decide whether or not it also has reduced row echelon form.

1. \[
\begin{bmatrix}
0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 \\
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}
\]
Given any matrix, we can always perform a sequence of elementary row operations to arrive at an equivalent matrix that has row echelon form. In fact, we can always perform a sequence of row operations to arrive at an equivalent matrix that has reduced row echelon form. For any non-zero matrix, there are infinitely many equivalent matrices that have row echelon form. However, there is only one equivalent matrix that has reduced row echelon form. (This is proved in Appendix A of the textbook, but we will not prove it in this course. We will just accept it to be true.)

**Theorem 2 (1)** Every matrix is equivalent to exactly one matrix that has reduced row echelon form.

**Example 3** Find infinitely many different matrices that have row echelon form and that are equivalent to the matrix

\[
\begin{bmatrix}
0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3
\end{bmatrix}
\]

Then find the unique matrix that has reduced row echelon form and that is equivalent to this matrix.

**Solution 4** By performing an interchange operation, we obtain

\[
\begin{bmatrix}
0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3
\end{bmatrix}
\sim
\begin{bmatrix}
0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The matrix on the right is equivalent to the matrix on the left and has row echelon form. If we like, we can now scale the top row by 3 to obtain the matrix

\[
\begin{bmatrix}
0 & 0 & 12 & -3 & 0 \\
0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
which also has row echelon form and which is also equivalent to the original matrix.

Clearly, we can obtain an infinite number of such matrices by continuing to scale the first or second rows by whatever non–zero number we like.

To find the unique reduced row echelon matrix that is equivalent to the original matrix, we continue the row operations started above as follows:

\[
\begin{bmatrix}
0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3
\end{bmatrix} \sim \begin{bmatrix}
0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
0 & 0 & 1 & -\frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
0 & 0 & 1 & -\frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

The matrix on the far right has reduced row echelon form and is equivalent to the original matrix. No other such reduced echelon matrix can be found. (This is guaranteed by Theorem 1.)

**Notation 5** For any matrix \( A \), we will use the notation \( \text{rref}(A) \) to denote the unique matrix having reduced row echelon form and equivalent to \( A \). (This notation is not used in the textbook we are using but it is done in many other books.)

For example, if

\[
A = \begin{bmatrix}
0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3
\end{bmatrix},
\]

then

\[
\text{rref}(A) = \begin{bmatrix}
0 & 0 & 1 & -\frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

### 1 Pivot Positions and Pivot Columns

**Definition 6** A **pivot position** in a matrix, \( A \), is a location in \( A \) that corresponds to a leading 1 in \( \text{rref}(A) \). A **pivot column** in \( A \) is a column of \( A \) that contains a pivot position.

For example, look at the matrices \( A \) and \( \text{rref}(A) \) in the example done above. The pivot positions in \( A \) are the positions indicated below:

\[
A = \begin{bmatrix}
0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3
\end{bmatrix}
\]

\[
\text{rref}(A) = \begin{bmatrix}
0 & 0 & 1 & -\frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & \overset{\text{pivot position}}{4} & -1 & 0 \\
0 & 0 & 0 & \overset{\text{pivot position}}{0} & 0 \\
0 & 0 & 0 & \overset{\text{pivot position}}{3}
\end{bmatrix}
\]
and the pivot columns of $A$ are the third and fifth columns. Note that we can only tell what the pivot positions and pivot columns of a matrix, $A$, are after we have found $rref(A)$.

**Example 7** Find the pivot positions and pivot columns of the matrix

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$ 

**Solution 8** Since

$$rref(A) = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the pivot positions of $A$ are as indicated below:

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}.$$ 

The pivot columns of $A$ are the first, second, and fourth columns.

## 2 The Row Reduction Algorithm

Here is an algorithm that always works for finding $rref(A)$ for any matrix $A$.

1. Begin with the leftmost non–zero column. This is a pivot column. The pivot position is at the top.

2. If the top entry in this column is 0, then interchange the top row with some other row that has a non–zero entry as its first entry. Then scale the top row to make the leading entry be a 1.
3. Use replacement operations to create zeros in every position in this column below the pivot position.

4. Cover (or ignore) the row containing the pivot position and cover (or ignore) any rows above it. Repeat steps 1, 2, and 3 for the submatrix that remains. Repeat the process until there is no non–zero column in the submatrix that remains.

5. Beginning with the rightmost pivot position and working upward and to the left, create zeros above each pivot position by using replacement operations.

Example 9 For the matrix

\[
A = \begin{bmatrix}
0 & 1 & -1 & 0 \\
2 & 0 & 5 & 5 \\
4 & 4 & 5 & 0
\end{bmatrix},
\]

use the reduction algorithm to find \( \text{rref}(A) \).

Answer:

\[
\text{rref}(A) = \begin{bmatrix}
1 & 0 & 0 & -\frac{45}{2} \\
0 & 1 & 0 & 10 \\
0 & 0 & 1 & 10
\end{bmatrix}.
\]

3 Solutions of Linear Systems

It is easy to find the solution set of a linear system whose augmented matrix has reduced row echelon form. Also, if \([A \mid b]\) is the augmented matrix of a system, then the solution set of this system is the same as the solution set of the system whose augmented matrix is \( \text{rref}([A \mid b]) \) (since the matrices \([A \mid b]\) and \( \text{rref}([A \mid b]) \) are equivalent). Thus, finding \( \text{rref}([A \mid b]) \) allows us to solve any given linear system. This is illustrated in the three examples that follow. In the first example, it turns out that the system is inconsistent. In the second example, the system has infinitely many solutions. In the third example, the system has a unique solution.
Example 10  Find the solution set of the linear system

\[
\begin{align*}
3x_1 - 4x_2 &= 9 \\
2x_1 + 4x_2 - x_3 &= 0 \\
10x_1 - 2x_3 &= -4.
\end{align*}
\]

Solution 11  The augmented matrix of this system is

\[
[A \mid b] = \begin{bmatrix} 3 & -4 & 0 & 9 \\ 2 & 4 & -1 & 0 \\ 10 & 0 & -2 & -4 \end{bmatrix}
\]

and

\[
\text{rref} ([A \mid b]) = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{3}{20} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

Since \( \text{rref} ([A \mid b]) \) is the augmented matrix of the linear system

\[
\begin{align*}
x_1 - \frac{1}{5}x_3 &= 0 \\
x_2 - \frac{3}{20}x_3 &= 0 \\
0 &= 1
\end{align*}
\]

which obviously has no solution (because of the equation \( 0 = 1 \)), we conclude that the original system that was given is also inconsistent. (Its solution set is the empty set.)

As an important remark, note that what causes this system to be inconsistent is the fact that the last column of its augmented matrix is a pivot column.

Example 12  Find the solution set of the linear system

\[
\begin{align*}
3x_1 - 4x_2 &= 9 \\
2x_1 + 4x_2 - x_3 &= 0 \\
10x_1 - 2x_3 &= 18.
\end{align*}
\]

Solution 13  The augmented matrix of this system is

\[
[A \mid b] = \begin{bmatrix} 3 & -4 & 0 & 9 \\ 2 & 4 & -1 & 0 \\ 10 & 0 & -2 & 18 \end{bmatrix}
\]
and

\[ \text{rref } ([A \mid b]) = \begin{bmatrix} 1 & 0 & -\frac{1}{5} & \frac{9}{5} \\ 0 & 1 & -\frac{3}{20} & -\frac{9}{10} \\ 0 & 0 & 0 & 0 \end{bmatrix} \].

The matrix \( \text{rref } ([A \mid b]) \) is the augmented matrix for the linear system

\[
\begin{align*}
    x_1 - \frac{1}{5}x_3 &= \frac{9}{5} \\
    x_2 - \frac{3}{20}x_3 &= -\frac{9}{10} \\
    0 &= 0
\end{align*}
\]

This system is consistent and has infinitely many solutions. To obtain a solution, we can let \( x_3 \) be any number that we like and then let

\[
x_2 = -\frac{9}{10} + \frac{3}{20}x_3,
\]

and then let

\[
x_1 = \frac{9}{5} + \frac{1}{5}x_3.
\]

Since \( x_3 \) can be any number that we like (for which reason we say that \( x_3 \) is a free variable), we see that the system under consideration has infinitely many solutions. As an example of a particular solution, suppose we let \( x_3 = 10 \). Then

\[
x_2 = -\frac{9}{10} + \frac{3}{20}(10) = \frac{3}{5}
\]

and

\[
x_1 = \frac{9}{5} + \frac{1}{5}(10) = \frac{19}{5}.
\]

Thus, the ordered triple \( \left( \frac{19}{5}, \frac{3}{5}, 10 \right) \) is a solution of the original system. Let us check:

\[
\begin{align*}
    3 \left( \frac{19}{5} \right) - 4 \left( \frac{3}{5} \right) &= 9 \\
    2 \left( \frac{19}{5} \right) + 4 \left( \frac{3}{5} \right) - 10 &= 0 \\
    10 \left( \frac{19}{5} \right) - 2 (10) &= 18.
\end{align*}
\]
As an important remark, note that was causes this system to have infinitely many solutions is the fact the last column of its augmented matrix is not a pivot column (thus making the system consistent) together with the fact that not every column of its coefficient matrix is a pivot column (allowing the system to have free variables).

**Example 14** Find the solution set of the linear system

\[
\begin{align*}
3x_1 - 4x_2 &= 9 \\
2x_1 + 4x_2 - x_3 &= 0 \\
x_1 - x_3 &= 0.
\end{align*}
\]

**Solution 15** The augmented matrix of this system is

\[
\begin{bmatrix}
A & | & b
\end{bmatrix} =
\begin{bmatrix}
3 & -4 & 0 & 9 \\
2 & 4 & -1 & 0 \\
1 & 0 & -1 & 0
\end{bmatrix}
\]

and

\[
\text{rref} \left( \begin{bmatrix}
A & | & b
\end{bmatrix} \right) =
\begin{bmatrix}
1 & 0 & 0 & \frac{9}{4} \\
0 & 1 & 0 & -\frac{9}{16} \\
0 & 0 & 1 & \frac{9}{4}
\end{bmatrix}.
\]

Since \( \text{rref} \left( \begin{bmatrix}
A & | & b
\end{bmatrix} \right) \) is the augmented matrix of the linear system

\[
\begin{align*}
x_1 &= \frac{9}{4} \\
x_2 &= -\frac{9}{16} \\
x_3 &= \frac{9}{4},
\end{align*}
\]

we conclude that the only solution of the original system is \( \left( \frac{9}{4}, -\frac{9}{16}, \frac{9}{4} \right) \).

Note that the reason that this system has a unique solution is that its last column is not a pivot column (meaning that the system is consistent) together with the fact that every column of its coefficient matrix is a pivot column (meaning that the system has no free variables).
4 Answering the “Existence and Uniqueness” Questions

Suppose that we have a linear system whose coefficient matrix is \( A \) and whose augmented matrix is \([A \mid b]\). Then \([A \mid b]\) is the same as \( A \) except for the fact that \([A \mid b]\) has an extra column on the right. Because of the way that elementary row operations work, the matrix \( \text{rref}([A \mid b]) \) is also the same as \( \text{rref}(A) \) except, once again, for the fact that \( \text{rref}([A \mid b]) \) has an extra column on the right. Inspired by the preceding three examples, we arrive at the following criteria for answering the questions:

1. Does our system have a solution? (Existence)

2. If so, then does it have just one solution or infinitely many? (Uniqueness)

**Theorem 16 (2)** For a linear system with coefficient matrix \( A \) and augmented matrix \([A \mid b]\):

1. The system is consistent if and only if the last column of \([A \mid b]\) is not a pivot column.

2. If the system is consistent, then the system has a unique solution if and only if every column of \( A \) is a pivot column.

**Example 17** Answer the existence and uniqueness questions for the system

\[
\begin{align*}
x_1 + 2x_2 - 5x_3 - 6x_4 &= -5 \\
x_2 - 6x_3 - 3x_4 &= 2 \\
x_4 - 5x_5 &= -5.
\end{align*}
\]

**Solution 18** The coefficient matrix for this system is

\[
A = \begin{bmatrix} 1 & 2 & -5 & -6 & 0 \\ 0 & 1 & -6 & -3 & 0 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}
\]
and the augmented matrix is

\[
[A \mid b] = \begin{bmatrix}
1 & 2 & -5 & -6 & 0 & -5 \\
0 & 1 & -6 & -3 & 0 & 2 \\
0 & 0 & 0 & 1 & -5 & -5
\end{bmatrix}.
\]

Also,

\[
\text{rref}([A \mid b]) = \begin{bmatrix}
1 & 0 & 7 & 0 & 0 & -9 \\
0 & 1 & -6 & 0 & -15 & -13 \\
0 & 0 & 0 & 1 & -5 & -5
\end{bmatrix}
\]

from which we can immediately conclude that the system is consistent (because the last column of \([A \mid b]\) is not a pivot column).

In addition,

\[
\text{rref}(A) = \begin{bmatrix}
1 & 0 & 7 & 0 & 0 \\
0 & 1 & -6 & 0 & -15 \\
0 & 0 & 0 & 1 & -5
\end{bmatrix},
\]

from which we conclude that the system has infinitely many solutions (rather than a unique solution) because not every column of \(A\) is a pivot column.

As an important remark, we note that it is not necessary to compute \(\text{rref}(A)\) separately once \(\text{rref}([A \mid b])\) has been computed. This is because \(\text{rref}(A)\) will always be the same as \(\text{rref}([A \mid b])\) with the last column deleted.

**Example 19** For the system given in the above example (which was found to have infinitely many solutions), give a parametric description of its solution set.

**Solution 20** Looking at the matrix \(\text{rref}([A \mid b])\) in the above example, we see that the system in that example is equivalent to the system

\[
\begin{align*}
x_1 + 7x_3 &= -9 \\
x_2 - 6x_3 - 15x_5 &= -13 \\
x_4 - 5x_5 &= -5.
\end{align*}
\]

The free variables for this system (corresponding to the non-pivot columns in \(\text{rref}(A)\)) are \(x_3\) and \(x_5\). The general solution of the system (described in
parametric form) is

\begin{align*}
x_1 &= -9 - 7x_3 \\
x_2 &= -13 + 6x_3 + 15x_5 \\
x_3 &= \text{free} \\
x_4 &= -5 + 5x_5 \\
x_5 &= \text{free}.
\end{align*}