Linear Transformations

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These notes closely follow the presentation of the material given in David C. Lay’s textbook *Linear Algebra and its Applications (3rd edition)*. These notes are intended primarily for in-class presentation and should not be regarded as a substitute for thoroughly reading the textbook itself and working through the exercises therein.

**Linear Transformations**

A transformation (or function or mapping) from \( V_n \) into \( V_m \) is an assignment of each vector \( x \in V_n \) to some vector in \( V_m \). For such a transformation \( T : V_n \rightarrow V_m \), we call \( V_n \) the **domain** of the transformation \( T \) and we call \( V_m \) the **codomain** of the transformation \( T \).

If the transformation \( T \) assigns the vector \( x \in V_n \) to the vector \( T(x) \in V_m \), then we call \( T(x) \) the **image** of \( x \) under the transformation \( T \). The set of all images of vectors in \( V_n \) under the transformation \( T \) is called the **range** of \( T \).

One way to define a transformation \( T : V_n \rightarrow V_m \) is via matrix multiplication. In particular, if \( A \) is an \( m \times n \) matrix, then \( T : V_n \rightarrow V_m \) defined by \( T(x) = Ax \) is a transformation. Sometimes when a transformation is defined via multiplication by a matrix \( A \), we describe this transformation by writing \( x \mapsto Ax \).
Example 1 Let $A$ be the matrix

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

and define the transformation $T : V_2 \to V_3$ by $T(x) = Ax$.

1. The domain of $T$ is ________.
2. The codomain of $T$ is ____________.
3. The image of the vector $u = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ under $T$ is ____________________.
4. Describe the range of $T$. 


Example 2 If $A$ is the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ can be visualized as a projection of points in $\mathbb{R}^3$ onto the $x_1, x_2$ plane.
Example 3 If $A$ is the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix},$$

then the transformation $x \mapsto Ax$ is called a shear.
Definition 4 A transformation $T : V_n \rightarrow V_m$ is called a linear transformation if

$$T(u + v) = T(u) + T(v) \text{ for all vectors } u \text{ and } v \text{ in } V_n$$

and

$$T(cu) = cT(u) \text{ for all vectors } u \text{ in } V_n \text{ and for all scalars } c.$$ 

Remark 5 If $A$ is an $m \times n$ matrix, then the transformation $T : V_n \rightarrow V_m$ defined by $T(x) = Ax$ is a linear transformation.

Some basic things that are true about any linear transformation $T : V_n \rightarrow V_m$ are:

1. $T(0_n) = 0_m$.

2. If $u$ and $v$ are any two vectors in $V_n$ and $c$ and $d$ are any two scalars, then

$$T(cu + dv) = cT(u) + dT(v).$$
Example 6  Given a scalar $r$, define $T : V_2 \rightarrow V_2$ by $T(x) = rx$. $T$ is called a contraction if $0 \leq r < 1$ and is called a dilation when $r > 1$.

Let $r = 3$ and show that $T$ is a linear transformation.
Example 7 Let the transformation \( T : V_2 \to V_2 \) be defined by

\[
T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}.
\]

1. Verify that \( T \) is a linear transformation.

2. Find the images of the vectors \( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \) and \( \begin{bmatrix} 0 \\ -1 \end{bmatrix} \) under \( T \).

3. Describe what the transformation \( T \) “does” to vectors \( \mathbf{x} \) in \( V_2 \).