Matrix Operations

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June 12, 2009

These notes closely follow the presentation of the material given in David C. Lay’s textbook Linear Algebra and its Applications (3rd edition). These notes are intended primarily for in-class presentation and should not be regarded as a substitute for thoroughly reading the textbook itself and working through the exercises therein.

Matrix Addition

If $A$ and $B$ are two matrices of the same size (both of size $m \times n$), then the sum of $A$ and $B$, denoted by $A + B$, is the $m \times n$ matrix whose entries are the sums of the corresponding entries of $A$ and $B$.

For example, if

$$A = \begin{bmatrix} 1 & -6 & 0 \\ 4 & 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & -9 & 4 \\ 4 & 1 & -4 \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} 1 & -15 & 4 \\ 8 & 3 & -2 \end{bmatrix}.$$ 

Multiplication of a Matrix by a Scalar

If $A$ is an $m \times n$ matrix and $r$ is a scalar, then the product of $r$ and $A$, denoted by $rA$, is the $m \times n$ matrix whose entries are the entries of $A$ multiplied by the scalar $r$. 
For example, if

$$A = \begin{bmatrix} 1 & -6 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

and $r = 4$,

then

$$rA = \begin{bmatrix} 4 & -24 & 0 \\ 16 & 8 & 8 \end{bmatrix}.$$ 

**Theorem 1** Let $A$, $B$, and $C$ be matrices of the same size and let $r$ and $s$ be scalars. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$ (where 0 is the matrix with the same size as $A$ and all of whose entries are 0)
4. $r (A + B) = rA + rB$
5. $(r + s) A = rA + sA$
6. $r (sA) = (rs) A$

**Multiplication of Two Matrices**

If $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix whose columns are $b_1, b_2, \ldots, b_p$, then the product $AB$ is the $m \times p$ matrix whose columns are $Ab_1, Ab_2, \ldots, Ab_p$.

For example, if

$$A = \begin{bmatrix} 1 & -6 & 0 \\ 4 & 2 & 2 \end{bmatrix}$$

and $B = \begin{bmatrix} 1 & 1 \\ -4 & 0 \\ 5 & -2 \end{bmatrix}$,
then

\[
Ab_1 = \begin{bmatrix} 1 & -6 & 0 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 25 \\ 6 \end{bmatrix}
\]

\[
Ab_2 = \begin{bmatrix} 1 & -6 & 0 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -8 \\ -2 \end{bmatrix} = \begin{bmatrix} 49 \\ -16 \end{bmatrix}
\]

\[
Ab_3 = \begin{bmatrix} 1 & -6 & 0 \\ 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},
\]

and so

\[
AB = \begin{bmatrix} Ab_1 & Ab_2 & Ab_3 \end{bmatrix} = \begin{bmatrix} 25 & 49 & 0 \\ 6 & -16 & 2 \end{bmatrix}.
\]
Why is The Matrix Product, $AB$, Defined In the Way That It Is?

Suppose that $A$ is an $m \times n$ matrix, $B = [ \ b_1 \ b_2 \ \cdots \ b_p \ ]$ is an $n \times p$ matrix, and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \in \mathbb{R}^p.$$ 

Then

$$Bx = [ \ b_1 \ b_2 \ \cdots \ b_p \ ] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = x_1 b_1 + x_2 b_2 + \cdots + x_p b_p,$$

and

$$A(Bx) = A(x_1 b_1 + x_2 b_2 + \cdots + x_p b_p)$$

$$= A(x_1 b_1) + A(x_2 b_2) + \cdots + A(x_p b_p)$$

$$= x_1 (Ab_1) + x_2 (Ab_2) + \cdots + x_p (Ab_p)$$

$$= \begin{bmatrix} Ab_1 \\ Ab_2 \\ \vdots \\ Ab_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} Ab_1 \\ Ab_2 \\ \vdots \\ Ab_p \end{bmatrix} x.$$ 

In summary, we see that

$$A(Bx) = \begin{bmatrix} Ab_1 \\ Ab_2 \\ \vdots \\ Ab_p \end{bmatrix} x.$$ 

We would like to define the matrix product $AB$ in such a way that the associative law

$$A(Bx) = (AB)x$$

will be true. In order for this associative law to be true, it must be true that

$$(AB)x = \begin{bmatrix} Ab_1 \\ Ab_2 \\ \vdots \\ Ab_p \end{bmatrix} x.$$
This will certainly be true if we define

\[ AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix} \]

and this is how we have defined it.
Two Other Ways to Compute $AB$

We will refer to the computation

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

as column–wise computation of $AB$.

Here are two other ways to compute $AB$.

Row–wise Computation of $AB$

If $A$ is an $m \times n$ matrix whose rows are $a_1, a_2, \ldots, a_m$, and $B$ is an $n \times p$ matrix, then the product $AB$ is the $m \times p$ matrix whose rows are $a_1B, a_2B, \ldots, a_mB$.

For example, if

$$A = \begin{bmatrix} 1 & -6 & 0 \\ 4 & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ -4 & -8 & 0 \\ 5 & -2 & 1 \end{bmatrix},$$

then

$$a_1B = \begin{bmatrix} 1 & -6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -4 & -8 & 0 \\ 5 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 25 \\ 49 \\ 0 \end{bmatrix}$$

$$a_2B = \begin{bmatrix} 4 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -4 & -8 & 0 \\ 5 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -16 \\ 2 \end{bmatrix}$$

and so

$$AB = \begin{bmatrix} a_1B \\ a_2B \end{bmatrix} = \begin{bmatrix} 25 & 49 & 0 \\ 6 & -16 & 2 \end{bmatrix}.$$  

Entry–wise Computation of $AB$

If $A$ is an $m \times n$ matrix whose rows are $a_1, a_2, \ldots, a_m$, and $B$ is an $n \times p$ matrix whose columns are $b_1, b_2, \ldots, b_p$, then $AB$ is the $m \times p$ matrix whose $(i, j)$ entry is $a_i b_j$. 

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For example, if
\[
A = \begin{bmatrix}
1 & -6 & 0 \\
4 & 2 & 2
\end{bmatrix}
\text{ and } B = \begin{bmatrix}
1 & 1 & 0 \\
-4 & -8 & 0 \\
5 & -2 & 1
\end{bmatrix},
\]
then
\[
AB = \begin{bmatrix}
a_1 b_1 & a_1 b_2 & a_1 b_3 \\
a_2 b_1 & a_2 b_2 & a_2 b_3
\end{bmatrix} = \begin{bmatrix}
25 & 49 & 0 \\
6 & -16 & 2
\end{bmatrix}.
\]

## Properties of Matrix Multiplication

Before stating a theorem that contains some important properties of matrix multiplication, we give a few more definitions.

If \(A\) is an \(m \times n\) matrix, then the transpose of \(A\), denoted by \(A^T\), is the \(n \times m\) matrix whose columns are the corresponding rows of \(A\).

For example, if
\[
A = \begin{bmatrix}
1 & -6 & 0 \\
4 & 2 & 2
\end{bmatrix},
\]
then
\[
A^T = \begin{bmatrix}
1 & 4 \\
-6 & 2 \\
0 & 2
\end{bmatrix}.
\]

The \(n \times n\) identity matrix, denoted by \(I_n\) (or just by \(I\) if the size of the matrix is clear from the context) is defined to be the \(n \times n\) matrix that has entries of 1 along its main diagonal and entries of 0 elsewhere.

For example the \(4 \times 4\) identity matrix is
\[
I_4 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

### Theorem 2
Suppose that \(A\), \(B\), and \(C\) are matrices of appropriate sizes such that the multiplications described below are all defined. Also, suppose that \(r\) is a scalar. Then
1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $IA = A = AI$ (Note that if $A$ is not a square matrix, then there are actually two different identity matrices involved in this statement. Specifically, if $A$ has size $m \times n$, then what this statement says is that $I_mA = A = AI_n$.)
6. $(A^T)^T = A$
7. $(A + B)^T = A^T + B^T$
8. $(rA)^T = rA^T$
9. $(AB)^T = B^TA^T$

We will prove statement 1 and part of statement 5 of this theorem. The rest will be left as homework.

**Proof of Statement 1:** Suppose that

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

is an $m \times n$ matrix, $B$ is an $n \times p$ matrix, and

$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_q \end{bmatrix}$$

is a $p \times q$ matrix.

Then, using row–wise computation, we see that

$$AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \begin{bmatrix} B \\ a_2B \\ \vdots \\ a_mB \end{bmatrix}$$
and, using entry–wise multiplication, we see that

\[(AB)C = \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_m B \end{bmatrix} \begin{bmatrix} c_1 & c_2 & \cdots & c_q \end{bmatrix} = \begin{bmatrix} a_1 B c_1 & a_1 B c_2 & \cdots & a_1 B c_q \\ a_2 B c_1 & a_2 B c_2 & \cdots & a_2 B c_q \\ \vdots & \vdots & \ddots & \vdots \\ a_m B c_1 & a_m B c_2 & \cdots & a_m B c_q \end{bmatrix} \]

We will now compute \( A(BC) \). Using column–wise computation, we obtain

\[ BC = B \begin{bmatrix} c_1 & c_2 & c_q \end{bmatrix} = \begin{bmatrix} B c_1 & B c_2 & B c_q \end{bmatrix} \]

and then using entry–wise computation, we obtain

\[ A(BC) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \begin{bmatrix} B c_1 & B c_2 & B c_q \end{bmatrix} = \begin{bmatrix} a_1 B c_1 & a_1 B c_2 & \cdots & a_1 B c_q \\ a_2 B c_1 & a_2 B c_2 & \cdots & a_2 B c_q \\ \vdots & \vdots & \ddots & \vdots \\ a_m B c_1 & a_m B c_2 & \cdots & a_m B c_q \end{bmatrix} \]

We now observe that \((AB)C = (AB)C\) and this completes the proof.

\[ \Box \]

**Proof that \( IA = A \):** Suppose that \( A \) is the \( m \times n \) matrix

\[ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \]
and let $I_m$ be the $m \times m$ identity matrix,

$$I_m = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}.$$  

Using entry–wise computation, we observe (after some verbal discussion and hand–waving) that

$$I_mA = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} = A$$

and this completes the proof. □

**Remark 3** Here is something to keep in mind about matrix multiplication: Even if both of the matrix products $AB$ and $BA$ are defined, they might not be equal. In other words, matrix multiplication does not have the commutative property.

For example,

$$\begin{bmatrix} 0 & -6 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} 36 & -24 \\ 57 & -30 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 0 & -6 \\ 3 & -9 \end{bmatrix} = \begin{bmatrix} 6 & -24 \\ 12 & 0 \end{bmatrix}.$$