The Square Matrix Theorem

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**Theorem 1 (The Square Matrix Theorem)** Suppose that $A$ is a square matrix of size $n \times n$. Then the following statements are all equivalent. That is, either all of these statements are true about the matrix $A$ or all of these statements are false about the matrix $A$.

1. Every column of $A$ has a pivot.
2. Every row of $A$ has a pivot.
3. $A \sim I_n$.
4. $A$ is invertible.
5. $\det(A) \neq 0$.
6. $A^T$ is invertible.
7. The set of column vectors of $A$ is linearly independent.
8. The set of row vectors of $A$ is linearly independent.
9. Any equation of the form $Ax = b$ (no matter what $b \in \mathbb{V}_n$ is) has a solution.
10. Any equation of the form $Ax = b$ (no matter what $b \in \mathbb{V}_n$ is) has a unique solution.
11. The homogeneous equation $Ax = 0_n$ has only the trivial solution.
12. The linear transformation $x \mapsto Ax$ is one–to–one.
13. The linear transformation $x \mapsto Ax$ is onto $V_n$.

14. The set of column vectors $A$ span $V_n$.

15. The set of row vectors $A$ span $V_n$.

There are also many other equivalent statements that could be added to the above theorem. For example, we could add the statement “The homogeneous equation $A^Tx = 0_n$ has only the trivial solution.”

**Example 2** Are the statements of the Square Matrix Theorem all true or all false about the matrix

$$A = \begin{bmatrix}
4 & -1 & -4 & -1 & -1 \\
-4 & -2 & 3 & 4 & 4 \\
-3 & -3 & 4 & -1 & 0 \\
5 & 1 & 5 & -2 & 3 \\
-5 & -5 & 1 & 5 & -2
\end{bmatrix}$$

**Example 3** Are the statements of the Square Matrix Theorem all true or all false about the matrix

$$A = \begin{bmatrix}
4 & -1 & 1 & -1 & -1 \\
-4 & -2 & 2 & 4 & 4 \\
-3 & -3 & -7 & -1 & 0 \\
5 & 1 & 7 & -2 & 3 \\
-5 & -5 & -7 & 5 & -2
\end{bmatrix}$$