Rank of a Matrix

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These notes closely follow the presentation of the material given in David C. Lay’s textbook *Linear Algebra and its Applications (3rd edition)*. These notes are intended primarily for in-class presentation and should not be regarded as a substitute for thoroughly reading the textbook itself and working through the exercises therein.

The Row Space of a Matrix

**Definition 1**  The *row space* of an $m \times n$ matrix, $A$, denoted by $\text{row} \ (A)$ is the set of all linear combinations of the row vectors of $A$.

**Definition 2**  The *column space* of an $m \times n$ matrix, $A$, denoted by $\text{col} \ (A)$ is the set of all linear combinations of the column vectors of $A$.

**Definition 3**  The *null space* of an $m \times n$ matrix, $A$, denoted by $\text{nul} \ (A)$ is the set of all solutions, $x$, of the equation $A x = 0_m$.

Here are some basic observations about the row space, column space, and null space:

1. If $A$ is an $m \times n$ matrix, then $\text{col} \ (A)$ is a subspace of $V_m$ and $\text{row} \ (A)$ is a subspace of $V_n$. In particular, $\text{col} \ (A)$ is the span of the columns of $A$ and $\text{row} \ (A)$ is the span of the rows of $A$.

2. If $A$ is an $m \times n$ matrix, then $\text{dim} \ (\text{col} \ (A)) + \text{dim} \ (\text{nul} \ (A)) = n$.

3. If $A$ is an $m \times n$ matrix, then $\text{row} \ (A) = \text{col} \ (A^T)$ and $\text{col} \ (A) = \text{row} \ (A^T)$. 

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Example 4 For the $4 \times 5$ matrix
\[
A = \begin{bmatrix}
-2 & -5 & 8 & 0 & -17 \\
1 & 3 & -5 & 1 & 5 \\
3 & 11 & -19 & 7 & 1 \\
1 & 7 & -13 & 5 & -3 \\
\end{bmatrix},
\]
find bases for $\text{col} (A)$, $\text{nul} (A)$, and $\text{row} (A)$.

Solution 5 From earlier work that we have done, we know how to find bases for $\text{col} (A)$ and $\text{nul} (A)$.

Since
\[
A = \begin{bmatrix}
-2 & -5 & 8 & 0 & -17 \\
1 & 3 & -5 & 1 & 5 \\
3 & 11 & -19 & 7 & 1 \\
1 & 7 & -13 & 5 & -3 \\
\end{bmatrix} - \begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]
we see that the first, second, and fourth columns of $A$ are the pivot columns of $A$. We also know that the pivot columns of $A$ form a basis for $\text{col} (A)$. Thus, $\text{col} (A)$ is three–dimensional and a basis for $\text{col} (A)$ is
\[
\left\{ \begin{bmatrix}
-2 \\
1 \\
3 \\
1 \\
\end{bmatrix}, \begin{bmatrix}
-5 \\
3 \\
11 \\
7 \\
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
7 \\
5 \\
\end{bmatrix} \right\}.
\]

To find a basis for $\text{nul} (A)$, we observe that every solution of $Ax = 0_4$ has the form
\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s - t \\ -3s + 2t \\ t \\ 5s \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 0 \\ 5 \end{bmatrix}.
\]

Therefore, $\text{nul} (A)$ is two–dimensional and a basis for $\text{nul} (A)$ is
\[
\left\{ \begin{bmatrix}
-1 \\
2 \\
1 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
5 \\
1 \\
\end{bmatrix} \right\}.
To find a basis for row \((A)\), we use the fact that \(\text{row}(A) = \text{col}(A^T)\). Observing that

\[
A^T = \begin{bmatrix}
-2 & 1 & 3 & 1 \\
-5 & 3 & 11 & 7 \\
8 & -5 & -19 & -13 \\
0 & 1 & 7 & 5 \\
-17 & 5 & 1 & -3
\end{bmatrix} - \begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 7 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

we see that the first, second, and fourth columns of \(A^T\) are the pivot columns of \(A^T\). Therefore, \(\text{col}(A^T) = \text{row}(A)\) is three-dimensional and a basis for \(\text{col}(A^T) = \text{row}(A)\) is

\[
\begin{bmatrix}
-2 \\
-5 \\
8 \\
0 \\
-17
\end{bmatrix}, \begin{bmatrix}
1 \\
3 \\
-5 \\
1 \\
5
\end{bmatrix}, \begin{bmatrix}
1 \\
7 \\
-13 \\
5 \\
-3
\end{bmatrix}.
\]
How to Find a Basis for $\text{col} (A)$ and $\text{row} (A)$ in One Fell Swoop

If $A$ is an $m \times n$ matrix, then every vector $v \in \text{col} (A)$ is a linear combination of the columns of $A$. This means that $v = Ax$ for some vector $x \in V_n$. Likewise, every vector $w \in \text{row} (A)$ is a linear combination of the rows of $A$. This means that $w = A^T y$ for some vector $y \in V_m$. We will use this way of looking at things to show that the row spaces of any two equivalent matrices are the same!

**Lemma 6** If $A$ and $B$ are $m \times n$ matrices and $A \sim B$, then there exists an invertible $m \times m$ matrix, $C$, such that $B = CA$. In particular, if $A$ is any $m \times n$ matrix, then there exists an invertible $m \times m$ matrix $C$ such that $\text{rref} (A) = CA$.

**Proof.** The fact that $A \sim B$ means that if we start with $A$ and perform a certain sequence of elementary row operations, then we arrive at $B$. This means that there is a sequence, $E_1, E_2, \ldots, E_k$ of elementary $m \times m$ matrices such that

$$E_k \ldots E_2 E_1 A = B.$$  

However, recall that all elementary matrices are invertible and recall that any product of invertible matrices is invertible. This means that the matrix

$$C = E_k \ldots E_2 E_1$$

is invertible.

We have shown that $B = CA$ where $C$ is an invertible $m \times m$ matrix. ■

**Example 7** As an example to illustrate the above lemma, let us consider the process of finding $\text{rref} (A)$ for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$
We do this, step by step, as follows:

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}
= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix}
= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}
= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}
= \text{rref}(A).
\]

In terms of elementary matrices, this procedure is written as

\[
\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix},
\]

which can be written as

\[
\begin{bmatrix} -\frac{5}{3} & \frac{2}{3} \\ \frac{4}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.
\]

Thus, in this example we see that \(\text{rref}(A) = CA\) where \(C\) is the matrix

\[
C = \begin{bmatrix} -\frac{5}{3} & \frac{2}{3} \\ \frac{4}{3} & -\frac{1}{3} \end{bmatrix}.
\]
Theorem 8 If $A$ and $B$ are $m \times n$ matrices and $A \sim B$, then $\text{row}(A) = \text{row}(B)$. In particular, if $A$ is any $m \times n$ matrix, then $\text{row}(A) = \text{row}(\text{rref}(A))$.

Proof. Since $A \sim B$, then we know that there exists an invertible $m \times m$ matrix $C$ such that $B = CA$.

We will now show that $\text{row}(B) \subseteq \text{row}(A)$.

Suppose that $w \in \text{row}(B)$. This means that there exists a vector $y \in V_m$ such that $w = B^T y$. From this, we obtain

$$w = (CA)^T y \quad \implies \quad w = (A^T C^T) y \quad \implies \quad w = A^T (C^T y).$$

Since the vector $C^T y$ is in $V_m$, the last equation above shows us that $w$ is a linear combination of the columns of $A^T$. This means that $w \in \text{row}(A)$. We have proved that $\text{row}(B) \subseteq \text{row}(A)$.

We will now show that $\text{row}(A) \subseteq \text{row}(B)$.

Suppose that $w \in \text{row}(A)$. This means that there exists a vector $y \in V_m$ such that $w = A^T y$. From this, we obtain

$$w = (C^{-1}B)^T y \quad \implies \quad w = \left(B^T (C^{-1})^T \right) y \quad \implies \quad w = B^T \left((C^{-1})^T y \right).$$

Since the vector $(C^{-1})^T y$ is in $V_m$, the last equation above shows us that $w$ is a linear combination of the columns of $B^T$. This means that $w \in \text{row}(B)$. We have proved that $\text{row}(A) \subseteq \text{row}(B)$.

Since $\text{row}(B) \subseteq \text{row}(A)$ and $\text{row}(A) \subseteq \text{row}(B)$, then $\text{row}(A) = \text{row}(B)$.

Theorem 9 If $A$ is an $m \times n$ matrix, then the non–zero rows of $\text{rref}(A)$ form a basis for $\text{row}(A)$ (and also form a basis for $\text{row}(\text{rref}(A))$ since $\text{row}(A) = \text{row}(\text{rref}(A))$).
Example 10  Use the above theorem to find a basis for the row space of the matrix

\[ A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}. \]

Solution 11  Since

\[ \text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]

we see that a basis for \( \text{row}(A) \) is

\[ \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \]

(Note that this basis for \( \text{row}(A) \) is different than the one that we found when we did this problem earlier using a different method.)
Example 12 Find bases for the column space, null space, and row space of the matrix

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}. \]

Solution 13 Since

\[ \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \]

we see that a basis for \( \text{col}(A) \) is

\[ \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}, \]

a basis for \( \text{nul}(A) \) is

\[ \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}, \]

and a basis for \( \text{row}(A) \) is

\[ \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}. \]
Remark 14 If $A$ and $B$ are $m \times n$ matrices and $A \sim B$, then it is not necessarily true that $\text{col}(A) = \text{col}(B)$. In particular, it is not necessarily true that $\text{col}(A) = \text{col}(\text{rref}(A))$. However, it is always true that $\dim(\text{col}(A)) = \dim(\text{col}(\text{rref}(A)))$.

Theorem 15 If $A$ is any $m \times n$ matrix, then the row space and the column space of $A$ have the same dimension.

Proof. The dimension of the column space of $A$ is the number of pivot columns of $\text{rref}(A)$. The dimension of the row space of $A$ is the number of non-zero rows of $\text{rref}(A)$. However, the number of pivot columns of $\text{rref}(A)$ is the same as the number of non-zero rows of $\text{rref}(A)$. Therefore, $\dim(\text{col}(A)) = \dim(\text{row}(A))$. \qed

Example 16 For the matrix
\[
A = \begin{bmatrix}
-2 & -5 & 8 & 0 & -17 \\
1 & 3 & -5 & 1 & 5 \\
3 & 11 & -19 & 7 & 1 \\
1 & 7 & -13 & 5 & -3
\end{bmatrix},
\]
we have seen that $\dim(\text{col}(A)) = \dim(\text{row}(A)) = 3$.

For the matrix
\[
A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix},
\]
we have seen that $\dim(\text{col}(A)) = \dim(\text{row}(A)) = 2$.

Definition 17 If $A$ is an $m \times n$ matrix, then we define the rank of $A$, denoted by $\text{rank}(A)$, to be the dimension of the column space of $A$ (which is the same as the dimension of the row space of $A$). If $A$ is a square matrix of size $n \times n$ and $\text{rank}(A) = n$, then we say that $A$ has full rank.

Theorem 18 An $n \times n$ matrix, $A$, is invertible if and only if $A$ has full rank.

Remark 19 Any of the many other statements (for example, $A^\top I_n$) that are given in the “Square Matrix Theorem” are equivalent to the statement that $A$ has full rank.