1. Consider the linear system

\[
\begin{align*}
2x + y &= 0 \\
x + y &= -1 \\
3x + y &= 1 \\
4x + y &= 2.
\end{align*}
\]

(a) Write down the coefficient matrix for this system.

**Answer:** The coefficient matrix is

\[
\begin{bmatrix}
2 & 1 \\
1 & 1 \\
3 & 1 \\
4 & 1
\end{bmatrix}.
\]

(b) Write down the augmented matrix for this system.

**Answer:** The augmented matrix is

\[
\begin{bmatrix}
2 & 1 & 0 \\
1 & 1 & -1 \\
3 & 1 & 1 \\
4 & 1 & 2
\end{bmatrix}.
\]

(c) Use Gauss-Jordan elimination to find the solution set of this system. In doing this, you must show your procedure **step-by-step.** Use the notation \( kR_i + R_j \rightarrow R_j \), etc., do describe what you are doing in each step.
Solution:

\[
\begin{bmatrix}
  2 & 1 & 0 \\
  1 & 1 & -1 \\
  3 & 1 & 1 \\
  4 & 1 & 2
\end{bmatrix}
\sim
\begin{bmatrix}
  1 & 1 & -1 \\
  0 & -1 & 2 \\
  3 & 1 & 1 \\
  4 & 1 & 2
\end{bmatrix}
\]

\[-2R_1 + R_2 \rightarrow R_2
\]

\[-3R_1 + R_3 \rightarrow R_3
\]

\[-4R_1 + R_4 \rightarrow R_4
\]

\[
\begin{bmatrix}
  1 & 1 & -1 \\
  0 & 1 & -2 \\
  0 & -2 & 4 \\
  0 & -3 & 6
\end{bmatrix}
\]

We can now see that the only solution of the system is \((x, y) = (1, -2)\)

(d) What is the solution set of this system?

**Answer:** The solution set is \(S = \{(1, -2)\}\).

2. Describe the solution set for the linear system corresponding to the reduced augmented matrix

\[
\begin{bmatrix}
  1 & 3 & -3 & 0 & 1 \\
  0 & 0 & 0 & 1 & 4
\end{bmatrix}
\]

**Answer:** The above augmented matrix corresponds to the system

\[
x_1 + 3x_2 - 3x_3 = 1
\]

\[x_4 = 4.
\]

We can thus see that all solutions of the system have the form

\[
x_4 = 4
\]

\[
x_3 = s
\]

\[
x_2 = t
\]

\[
x_1 = 1 - 3t + 3s.
\]

The solution set of the system is

\[
S = \{(x_1, x_2, x_3, x_4) = (1 - 3t + 3s, t, s, 4) \mid s \in R, t \in R\}.
\]
3. Let $A$ and $B$ be the matrices

$$A = \begin{bmatrix} 1 & -2 \\ x & 0 \end{bmatrix}, \quad B = \begin{bmatrix} x & 0 \\ x & 1 \end{bmatrix}.$$ 

(a) Compute $AB$ and $BA$. (Show the details of how you do this.)

**Solution:**

$$AB = \begin{bmatrix} 1 & -2 \\ x & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ x & 1 \end{bmatrix} = \begin{bmatrix} x - 2x & 0 - 2 \\ x^2 + 0 & 0 + 0 \end{bmatrix} = \begin{bmatrix} -x & -2 \\ x^2 & 0 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} x & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ x & 0 \end{bmatrix} = \begin{bmatrix} x + 0 & -2x + 0 \\ x + x & -2x + 0 \end{bmatrix} = \begin{bmatrix} x & -2x \\ 2x & -2x \end{bmatrix}.$$ 

(b) Are there any values of $x$ that would make $AB = BA$? Explain.

**Solution:** There are no values of $x$ that would make $AB = BA$. If there were, then we would have to have $x = -x$ which would imply that $x = 0$. However it is easy to check that $x = 0$ does not work. For example $-2 \neq -2 \cdot 0$.

4. Use the algorithm for computing inverses to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & -1 \\ 2 & -2 & 1 \end{bmatrix}.$$ 

All details must be included in order to get credit.

**Solution:**

$$[A \mid I] = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ -2 & 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 2 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{bmatrix} = [I \mid A^{-1}]$$

Thus

$$A^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 2 & 0 & -1 \end{bmatrix}.$$
5. Consider the linear system
\[\begin{align*}
2x + y &= 1 \\
-x + y &= -2 \\
x + 2y &= -1.
\end{align*}\]

(a) Write this system as a matrix equation, \(Ax = b\).

**Answer:** We can write the system as the matrix equation
\[
\begin{bmatrix}
2 & 1 \\
-1 & 1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
1 \\
-2 \\
-1
\end{bmatrix}.
\]

(b) Find the solution of this system.

**Solution:** By row-reducing the augmented matrix
\[
\begin{bmatrix}
2 & 1 & 1 \\
-1 & 1 & -2 \\
1 & 2 & -1
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix},
\]
we see that the only solution of this system is \((x, y) = (1, -1)\).

(c) Let \(C\) be the matrix
\[
C = \begin{bmatrix}
\frac{2}{3} & 0 & -\frac{1}{3} \\
\frac{2}{3} & 1 & -\frac{2}{3}
\end{bmatrix}
\]
Show that \(CA = I_2\). (For this reason, \(C\), is called a left inverse of \(A\).)

**Solution:**
\[
CA = \begin{bmatrix}
\frac{2}{3} & 0 & -\frac{1}{3} \\
\frac{2}{3} & 1 & -\frac{2}{3}
\end{bmatrix}
\begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = I_2.
\]

(d) Show that the solution to the linear system is given by \(x = Cb\).

**Solution:** Since \(CA = I_2\), we have that if \(Ax = b\), then \(C(Ax) = Cb\) which implies that \((CA)x = Cb\) which implies that \(I_2x = Cb\) which implies that \(x = Cb\). We can also show this by direct computation:
\[
Cb = \begin{bmatrix}
\frac{2}{3} & 0 & -\frac{1}{3} \\
\frac{2}{3} & 1 & -\frac{2}{3}
\end{bmatrix}
\begin{bmatrix}
1 \\
-2 \\
-1
\end{bmatrix} =
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
\]
which is the same answer we obtained in part b.

6. Let
\[
A = \begin{bmatrix}
-3 & 1 & 1 \\
-3 & -3 & -2 \\
3 & -1 & -1
\end{bmatrix}.
\]

Explain how we can tell very quickly, without actually going through the process of computing \(\det(A)\), that \(\det(A) = 0\). (Write your answer in complete sentences.)
If you can’t see how to do this without doing computations, then do the computations to show that \( \det(A) = 0 \). (You will get credit either way.)

**Answer:** Since the third row of \( A \) is a scalar multiple of the first row \( (R_3 = -1 \cdot R_1) \), then \( \det(A) = 0 \).

7. Prove that if \( A \) is a 2 \times 2 matrix and \( \det(A) = 0 \), then \( A \) is not invertible.

**Answer:** See your notes.

8. Decide whether each of the following statements is true or false. Grading of this question will be as follows:

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<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
</tr>
</tbody>
</table>

(a) Any system of linear equations has either no solutions, exactly one solution, or infinitely many solutions. (True / False)

(b) The matrix

\[
\begin{bmatrix}
1 & -6 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

has reduced row–echelon form. (True / False)

(c) It is possible for a homogeneous linear system, \( A\mathbf{x} = \mathbf{0} \), to be inconsistent. (True / False)

(d) If \( A \) is an \( n \times n \) matrix and the homogeneous system \( A\mathbf{x} = \mathbf{0} \) has non–trivial solutions, then \( A \) is not invertible. (True / False)

(e) If \( A \) is an invertible \( n \times n \) matrix, then it is possible to choose a vector \( \mathbf{b} \) such that \( A\mathbf{x} = \mathbf{b} \) has no solutions and it is also possible to choose a vector \( \mathbf{b} \) such that \( A\mathbf{x} = \mathbf{b} \) has infinitely many solutions. (True / False)

(f) If \( A \) and \( B \) are any two square matrices of the same size, then \( AB = BA \). (True / False)

(g) If \( B \) is a matrix that is obtained from a matrix \( A \) by interchanging two rows of \( A \), then \( \det(B) = -\det(A) \). (True / False)

(h) If \( B \) is a matrix that is obtained from a matrix \( A \) by multiplying a single row of \( A \) by a scalar, \( c \), then \( \det(B) = c\det(A) \). (True / False)

(i) If \( A \) is a square matrix and \( \text{rref}(A) = I \), then \( A\mathbf{x} = \mathbf{b} \) has a unique solution (for any choice of the vector \( \mathbf{b} \)). (True / False)

(j) If \( A \) and \( B \) are \( n \times n \) matrices and both \( A \) and \( B \) are invertible, then \( AB \) is invertible. (True / False)