Definition 1  An \( n \times n \) matrix, \( A \), is said to be \textbf{invertible} if there exists an \( n \times n \) matrix \( B \) such that \( AB = BA = I_n \) (where \( I_n \) is the \( n \times n \) identity matrix).

Remark 2  We know that if \( A \) has an inverse, then that inverse is unique. Thus we denote the inverse of \( A \) by \( A^{-1} \).

Definition 3  If
\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
is a \( 2 \times 2 \) matrix, then we define the \textit{determinant} of \( A \), denoted either by \( \det (A) \) or \( |A| \), to be
\[
\det (A) = ad - bc.
\]

Theorem 4  Suppose that
\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]
is a \( 2 \times 2 \) matrix.

1. If \( \det (A) \neq 0 \), then \( A \) is invertible and
\[
A^{-1} = \frac{1}{\det (A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

2. If \( \det (A) = 0 \), then \( A \) is not invertible.
Proof. Suppose that \( \det (A) \neq 0 \) and let

\[
B = \frac{1}{\det (A)} \begin{bmatrix}
d & -b \\
- c & a
\end{bmatrix}.
\]

Then, by direct computation, we have

\[
AB = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \left( \frac{1}{\det (A)} \begin{bmatrix}
d & -b \\
- c & a
\end{bmatrix} \right)
= \frac{1}{\det (A)} \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
d & -b \\
- c & a
\end{bmatrix}
= \frac{1}{ad - bc} \begin{bmatrix}
0 & ad - bc \\
0 & ad - bc
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = I_2.
\]

A similar computation shows that it is also true that \( BA = I_2 \). Therefore \( A \) is invertible and \( A^{-1} = B \) (as defined above).

Now let us assume that \( \det (A) = 0 \). Then

\[
ad - bc = 0. \tag{1}
\]

For the sake of obtaining a contradiction, let us now suppose that \( A \) is invertible. Then there is a \( 2 \times 2 \) matrix,

\[
B = \begin{bmatrix}
x_1 & x_2 \\
x_3 & x_4
\end{bmatrix},
\]

such that \( AB = I_2 \). Thus

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
x_1 & x_2 \\
x_3 & x_4
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

By performing the above matrix multiplication, we see that it must then be true that

\[
a x_1 + b x_3 = 1 \tag{2}
\]
\[
a x_2 + b x_4 = 0 \tag{3}
\]
\[
c x_1 + d x_3 = 0 \tag{4}
\]
\[
c x_2 + d x_4 = 1. \tag{5}
\]
We will now consider two cases:

**Case 1:** Suppose that $a = 0$. Then, by equation (2), $b \neq 0$. Consequently, by equation (1), $c = 0$ and, by equation (3), $x_4 = 0$. However, this means that equation (5) is not satisfied (because $c = 0$ and $x_4 = 0$). Thus it cannot be the case that $a = 0$.

**Case 2:** Suppose that $a \neq 0$. Then by performing the elementary operation

$$\frac{c}{a} E_1 + E_3 \rightarrow E_3,$$

we obtain the system

$$ax_1 + bx_3 = 1$$

$$\left(\frac{bc}{a} + d\right)x_3 = -\frac{c}{a}$$

which can also be written as

$$ax_1 + bx_3 = 1$$

$$\frac{ad - bc}{a} x_3 = -\frac{c}{a}.$$  

Since $ad - bc = 0$, then it must be the case that $c = 0$. But then we must also have $d = 0$ by equation (1). However, this means that equation (1) is not satisfied, so we have once again arrived at a contradiction.

We conclude that if $\det(A) = 0$, then $A$ is not invertible. ■

**Theorem 5** Suppose that $A$ and $B$ are $2 \times 2$ matrices. Then

$$\det(AB) = \det(A) \det(B).$$

**Proof.** We will prove this by computation. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}. $$

Then

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$
and thus

$$\det(AB) = (ae + bg)(cf + dh) - (af + bh)(ce + dg)$$
$$= acef + adeh + bcfg + bdgh$$
$$- acef - adfg - bceh - bdgh$$
$$= adeh + bcfg - adfg - bceh.$$ 

Also,

$$\det(A) \det(B) = (ad - bc)(eh - fg) = adeh - adfg - bceh + bcfg.$$ 

This shows that $\det(AB) = \det(A) \det(B)$. $\blacksquare$