The Inverse of a Square Matrix

The $n \times n$ identity matrix is the $n \times n$ matrix that has entries of 1 all along its main diagonal and entries of 0 elsewhere. We denote this matrix by $I_n$ or just by $I$ if it is clear what $n$ is. For example,

$$ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} $$

and

$$ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. $$

The matrix $I_n$ acts as a multiplicative identity for matrix multiplication because if $A$ is any $n \times n$ matrix, then

$$ AI_n = I_n A = A. $$
Definition of the Inverse of a Square Matrix

Suppose that $A$ is an $n \times n$ matrix. If there exists an $n \times n$ matrix, $B$ such that $AB = BA = I_n$, then $B$ is said to be a **multiplicative inverse** of $A$.

Example

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and verify that

$$B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

is a multiplicative inverse of $A$. 
Theorem (Uniqueness of Inverses)

If $B$ and $C$ are both inverses of a given matrix $A$, then $B = C$. In other words, the inverse of a matrix (if it exists) is unique.

Proof

(to be given in class).

Invertibility

If a matrix, $A$, has an inverse, then we say that $A$ is invertible. If $A$ is invertible, then we know that it has a unique inverse. Thus we can unambiguously use the notation $A^{-1}$ to denote the inverse of an invertible matrix $A$. 
The Inverse of a $2 \times 2$ Matrix

Let $A$ be the $2 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$  

1) Show that if $ad - bc \neq 0$, then the inverse of $A$ is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$  

2) Show that if $ad - bc = 0$, then $A$ is not invertible.
Inverses of Larger Matrices

To motivate our method for finding the inverse of square matrices whose size is larger than $2 \times 2$, let us find the inverse of the $3 \times 3$ matrix

$$A = \begin{bmatrix}
1 & 1 & -2 \\
-1 & 2 & 0 \\
0 & -1 & 1
\end{bmatrix}.$$ 

To do this we let

$$A^{-1} = \begin{bmatrix}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33}
\end{bmatrix}$$

and then solve $AA^{-1} = I_3$ or

$$\begin{bmatrix}
1 & 1 & -2 \\
-1 & 2 & 0 \\
0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23} \\
x_{31} & x_{32} & x_{33}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.$$
General Procedure for Finding Inverses

Suppose that \( A \) is an \( n \times n \) matrix and we would like to find \( A^{-1} \) (if it exists). If write the unknown matrix, \( A^{-1} \), as

\[
A^{-1} = [x_1 \ x_2 \ \cdots \ x_n]
\]

where \( x_1, x_2, \ldots, x_n \) are the column vectors of \( A^{-1} \) and write \( I_n \) as

\[
I_n = [e_1 \ e_2 \ \cdots \ e_n]
\]

(where \( e_i \) is the vector with a 1 in position \( i \) and zeros elsewhere), we can then observe that all \( n \) of the equations

\[
Ax_i = e_i
\]

must be satisfied.

Since it is a waste of effort to solve each one of these \( n \) equations separately, we do it in one fell swoop by row reducing the augmented matrix \([A \ | \ I_n]\). Once this is done, we will have an augmented matrix of the form \([\text{rref}(A) \ | \ B]\). If \( \text{rref}(A) = I_n \), then \( B = A^{-1} \). If \( \text{rref}(A) \neq I_n \), then \( A \) is not invertible.
Example

Use the algorithm described above to find the inverse (if it exists) of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

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Theorem (Products of Invertible Matrices)

1) If $A$ is an invertible $n \times n$ matrix, then $A^{-1}$ is also invertible and $(A^{-1})^{-1} = A$.

2) If $A$ and $B$ are both invertible $n \times n$ matrices, then $AB$ is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$ 

3) If either one (or both) of $A$ and $B$ is not invertible, then $AB$ is not invertible.

Proof

(to be given in class)
Homework

In Section 1.4 (page 45), do problems 1–29 (odd).