1.6: Determinants

The Determinant of a $2 \times 2$ Matrix

The determinant of a $2 \times 2$ matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

is denoted by $\det (A)$ or by $|A|$ and is defined as

$$|A| = ad - bc.$$

Example

Compute the determinant of the $2 \times 2$ matrix

$$A = \begin{bmatrix} -1 & 4 \\ -4 & 1 \end{bmatrix}.$$
The Determinant of a $3 \times 3$ Matrix

The determinant of a $3 \times 3$ matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

is defined to be $|A| =$

$$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example

Compute the determinant of the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -2 & -3 & 1 \\ 1 & -3 & -3 \end{bmatrix}.$$
General Definition of Determinant

If $A$ is an $n \times n$ matrix, then the minor, $M_{ij}$, associated with the entry $a_{ij}$ is the determinant of the $(n - 1) \times (n - 1)$ matrix obtained by deleting row $i$ and column $j$ of $A$. The cofactor associated with $a_{ij}$ is $C_{ij} = (-1)^{i+j} M_{ij}$. The determinant of $A$ is defined by

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$  

Example

Find the minors and cofactors of the $4 \times 4$ matrix

$$A = \begin{bmatrix} 2 & -4 & 1 & -4 \\ -1 & -1 & 3 & 2 \\ 0 & -1 & 0 & 0 \\ 4 & 1 & 2 & 4 \end{bmatrix}$$

and then use these to compute the determinant of $A$. 

Some Useful Facts

1. The determinant of a matrix, $A$, can be computed by cofactor expansion along any row or column of $A$. (We will not prove this in this course.)

2. The determinant of a triangular matrix (having either all zeros above or all zeros below the main diagonal) is the product of the entries on the main diagonal.

Example

Let

$$A = \begin{bmatrix}
2 & -4 & 1 & -4 \\
-1 & -1 & 3 & 2 \\
0 & -1 & 0 & 0 \\
4 & 1 & 2 & 4
\end{bmatrix}.$$  

What is a good choice of row or column along which to perform a cofactor expansion of $A$? Compute the determinant of $A$ by cofactor expansion along this row or column.
Example

Compute the determinant of the triangular matrix

\[ A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
2 & -2 & 0 & 0 & 0 \\
0 & 2 & 3 & 0 & 0 \\
-3 & 1 & 1 & 2 & 0 \\
0 & -4 & 0 & -4 & 2
\end{bmatrix} \]
Theorem: Effects of Row Operations on Determinants

Let $A$ be a square matrix.

1. If two rows of $A$ are interchanged to produce matrix $B$, then $\det(B) = -\det(A)$.

2. If a row of matrix $A$ is multiplied by a scalar, $c$, to produce a matrix $B$, then $\det(B) = c \det(A)$.

3. If $c$ is a scalar and row $R_j$ of matrix $A$ is replaced by $cR_i + R_j$ to produce a matrix $B$, then $\det(B) = \det(A)$. 
Example

Let

\[
A = \begin{bmatrix}
0 & 1 & 3 & -1 \\
2 & 4 & -6 & 1 \\
0 & 3 & 9 & 2 \\
0 & 0 & -5 & -2
\end{bmatrix}.
\]

Use elementary row operations to assist in computing \(|A|\).
Theorem: Properties of Determinants

Let $A$ and $B$ be $n \times n$ matrices and let $c$ be a scalar.

1) $\det(AB) = \det(A) \det(B)$.

2) $\det(cA) = c^n \det(A)$.

3) $\det(A^t) = \det(A)$.

4) If $A$ has a row or a column of zeros, then $\det(A) = 0$.

5) If $A$ has a row (column) that is a multiple of another row (column), then $\det(A) = 0$. 
Theorem

A square matrix, \( A \), is invertible if and only if \( \det (A) \neq 0 \).

Corollary

If \( A \) is an invertible square matrix, then

\[
\det \left( A^{-1} \right) = \frac{1}{\det (A)}.
\]
Theorem

Let $A$ be an $n \times n$ matrix. Then the following statements are equivalent (meaning that they are either all true or all false).

1) $A$ is invertible.

2) The homogeneous linear system $Ax = 0$ has only the trivial solution.

3) For any given $n$–dimensional vector, $b$, the linear system $Ax = b$ has a unique solution.

4) $\text{rref}(A) = I_n$.

5) $\det(A) \neq 0$. 
Homework

In Section 1.6 (page 65), do problems 1–37 (odd).