2.1: Vectors

For any positive integer \( n \), the set of all \( n \times 1 \) matrices (with real numbers as entries) is called \( R^n \). Each matrix in \( R^n \) is called an \( n \)-vector. Vectors are usually denoted by boldface letters, such as \( \mathbf{v} \), or by letters with an arrow above, such as \( \vec{v} \).

For example,

\[
\mathbf{v} = \begin{bmatrix}
-3 \\
5
\end{bmatrix}
\quad \text{and} \quad
\mathbf{w} = \begin{bmatrix}
0 \\
6.75
\end{bmatrix}
\]

are vectors in \( R^2 \), and

\[
\mathbf{v} = \begin{bmatrix}
1 \\
-5 \\
-5
\end{bmatrix}
\quad \text{and} \quad
\mathbf{w} = \begin{bmatrix}
9 \\
\sqrt{3} \\
-4.3
\end{bmatrix}
\]

are vectors in \( R^3 \).

In general, a vector in \( R^n \) has the form

\[
\mathbf{v} = \begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix}.
\]

The Algebra of Vectors

For vectors

\[
\mathbf{v} = \begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix}
\quad \text{and} \quad
\mathbf{w} = \begin{bmatrix}
w_1 \\
w_2 \\
\vdots \\
w_n
\end{bmatrix}
\]
in \( R^n \), the sum of \( \mathbf{v} \) and \( \mathbf{w} \) is defined to be the vector

\[
\mathbf{v} + \mathbf{w} = \begin{bmatrix}
v_1 + w_1 \\
v_2 + w_2 \\
\vdots \\
v_n + w_n
\end{bmatrix}.
\]
and the difference, \( \mathbf{v} - \mathbf{w} \) is defined to be the vector

\[
\mathbf{v} - \mathbf{w} = \begin{bmatrix}
v_1 - w_1 \\
v_2 - w_2 \\
\vdots \\
v_n - w_n
\end{bmatrix}.
\]

For a real number (also called a \textit{scalar}) \( c \), the \textit{scalar multiple} of the vector \( \mathbf{v} \) by the scalar \( c \) is defined to be the vector

\[
c\mathbf{v} = \begin{bmatrix}
cv_1 \\
cv_2 \\
\vdots \\
cv_n
\end{bmatrix}.
\]

For example, suppose that

\[
\mathbf{v} = \begin{bmatrix} 1 \\ -5 \\ -5 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 9 \\ \sqrt{3} \\ -4.3 \end{bmatrix}.
\]

Then

\[
\mathbf{v} + \mathbf{w} = \begin{bmatrix} 10 \\ -5 + \sqrt{3} \\ -9.3 \end{bmatrix}
\]

and

\[
6\mathbf{v} = \begin{bmatrix} 6 \\ -30 \\ -30 \end{bmatrix}.
\]
Additive Inverses and Zero Vectors

For any vector

\[ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \]

the **opposite** of \( \mathbf{v} \) (also called the **negative** of \( \mathbf{v} \) or the **additive inverse** of \( \mathbf{v} \)) is defined to be the vector

\[ -\mathbf{v} = \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix}. \]

Also, the vector in \( \mathbb{R}^n \) that has zero as all of its entries is called the **zero vector** in \( \mathbb{R}^n \) and is denoted by \( \mathbf{0}_n \) or simply by \( \mathbf{0} \) if the context is clear. Thus, for example,

\[ \mathbf{0}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

is the zero vector in \( \mathbb{R}^3 \) and if no confusion should arise, then we could simply write

\[ \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \]
Geometric Interpretations of Vector Addition and Scalar Multiplication in $\mathbb{R}^2$

We visualize a vector

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

in $\mathbb{R}^2$ to be an arrow with base at the origin, (0, 0), and with tip at the point $(x, y)$. For example, the vector

$$\mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

is visualized as in the figure below:

![Vector Visualization](image)

The zero vector, 0, is visualized as a point at the origin.
If $v$ and $w$ are vectors in $R^2$, then $v + w$ can be visualized as the arrow with base at the origin and with tip at the fourth vertex of the parallelogram whose other three vertices are at the origin, the tip of $v$, and the tip of $w$. For example, if

$$v = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

then

$$v + w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

is visualized as shown below.
A scalar multiple of a vector, \( cv \), is visualized as a vector that lies along the same line as \( v \) and points either in the same direction (if \( c > 0 \)) or the opposite direction (if \( c < 0 \)) as \( v \). Also, \( cv \) is \( c \) times as long as \( v \). For example,

\[
v = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \quad \text{and} \quad -2v = \begin{bmatrix} 4 \\ -6 \end{bmatrix}
\]

are pictured below:

A couple of final remarks about this:
Since \( -v = -1v \), then \( -v \) points in the opposite direction of \( v \) and has the same length as \( v \).
Since \( v - w = v + (-w) \), then the geometric picture of vector subtraction can be looked at in the same way as vector addition.
Geometric Interpretations of Vector Addition and Scalar Multiplication in $R^3$

Since any two vectors in $R^3$ actually lie in a two-dimensional plane, the geometric pictures of vector addition and scalar multiplication in $R^3$ can be pictured the same way as in $R^2$ (although it is harder to draw three-dimensional pictures).
Algebraic Properties of Vectors in $\mathbb{R}^n$

For all vectors $u, v,$ and $w$ in $\mathbb{R}^n$ and for all scalars $c$ and $d$:

1. $u + v = v + u$
2. $u + (v + w) = (u + v) + w$
3. $u + 0 = u$
4. $u + (-u) = 0$
5. $c (u + v) = cu + cv$
6. $(c + d) u = cu + du$
7. $c (du) = (cd) u$
8. $1u = u$