Some Linear Algebra

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Abstract

We review some linear algebra that is useful to know in studying linear systems of differential equations.

1 Solutions of The Equation $Ax = b$

Consider the following system of linear equations in unknowns $x$ and $y$:

\[
2x - 5y = 9 \\
3x + 4y = 2.
\]

There are two other ways to write this system: We can write it as a vector equation

\[
x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix} \]

(2)

or as a matrix equation

\[
\begin{bmatrix} 2 & -5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}.
\]

(3)

We can find the solution set of the linear system of equations (1) by using some basic algebra: If $(x, y)$ is a solution of the system

\[
2x - 5y = 9 \\
3x + 4y = 2,
\]

We can solve for $x$ and $y$:
then

\[ 6x - 15y = 27 \]
\[ -6x - 8y = -4 \]

and (by adding these two equations)

\[ -23y = 23 \]

which shows that \( y = -1 \). Using the first of the equations in the system, we obtain

\[ 2x - 5 (-1) = 9 \]

which gives us \( x = 2 \). We conclude that the solution of the linear system (1) is \( (x, y) = (2, -1) \). This pair, \( (x, y) = (2, -1) \), is also the solution of the vector equation (2) and the matrix equation (3). To see that \( (x, y) = (2, -1) \) satisfies all three equations, observe that

\[
\begin{align*}
2(2) - 5(-1) &= 9 \\
3(2) + 4(-1) &= 2
\end{align*}
\]

and

\[
\begin{align*}
(2) \begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} -5 \\ 4 \end{bmatrix} &= \begin{bmatrix} 9 \\ 2 \end{bmatrix} \\
\begin{bmatrix} 2 & -5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} &= \begin{bmatrix} 9 \\ 2 \end{bmatrix}
\end{align*}
\]

In general, if we have any system of linear equations in two unknowns

\[
\begin{align*}
a_{11}x + a_{12}y &= b_1 \\
a_{21}x + a_{22}y &= b_2,
\end{align*}
\]

then we can write this system as the vector equation

\[
x \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + y \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\]

or as the matrix equation

\[
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.
\]
System (4), equation (5) and equation (6) are all equivalent to one another, meaning that they all have the same solution set. The solution set of any linear system of equations consists of either exactly one solution or infinitely many solutions or no solutions. In the case that there is exactly one solution, we say that the system has a unique solution. The system (1) is an example of a linear system with a unique solution.

Exercise 1 How many solutions does the linear system
\[
\begin{align*}
4x - 3y &= 3 \\
-8x + 6y &= -6
\end{align*}
\]
have? Describe the solution set of this system. Do the same for the linear system
\[
\begin{align*}
4x - 3y &= 3 \\
-8x + 6y &= -3
\end{align*}
\]

A linear system of the form
\[
\begin{align*}
a_{11}x + a_{12}y &= 0 \\
a_{21}x + a_{22}y &= 0
\end{align*}
\]  \hspace{1cm} (7)
is called a homogeneous system. A homogeneous system always has at least one solution. In particular, it is easily seen that \((x, y) = (0, 0)\) is a solution. The solution \((x, y) = (0, 0)\) is called the trivial solution of the homogeneous system (7). If the system has solutions other than \((x, y) = (0, 0)\), then these other solutions are called non-trivial solutions.

Exercise 2 Decide whether each of the following homogeneous systems has non-trivial solutions. Describe the solution set of each system.

1.
\[
\begin{align*}
2x - 5y &= 0 \\
3x + 4y &= 0.
\end{align*}
\]

2.
\[
\begin{align*}
4x - 3y &= 0 \\
-8x + 6y &= 0.
\end{align*}
\]
We will now state a theorem that gives several equivalent criteria for determining information about the solution set of a linear system. Before doing this, let us make the convention that for a linear system

\begin{align*}
    a_{11}x + a_{12}y &= b_1, \\
    a_{21}x + a_{22}y &= b_2,
\end{align*}

we will define the vectors \( \mathbf{v}_1, \mathbf{v}_2 \), and \( \mathbf{b} \) by

\[
\mathbf{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\]

so that the vector equation that is equivalent to system (8) can be written as

\[
x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{b}.
\]

The notation \( \mathbf{0} \) is used to denote the zero vector

\[
\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Let us also make the convention to define the matrix \( A \) by

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.
\]

The matrix \( A \) is called the coefficient matrix of the system (8).

The (unknown) vector \( \mathbf{x} \) is defined by

\[
\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}
\]

so that the equivalent matrix equation that is equivalent to system (8) can be written as

\[
A\mathbf{x} = \mathbf{b}.
\]

Finally, we give two important definitions that are relevant to the theorem that will be stated:

**Definition 3** The set of vectors \( \{\mathbf{v}_1, \mathbf{v}_2\} \) is said to be a **linearly independent** set of vectors if the homogeneous vector equation

\[
x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{0}
\]

has only the trivial solution. Otherwise, the set of vectors \( \{\mathbf{v}_1, \mathbf{v}_2\} \) is said to be a **linearly dependent** set of vectors.
Definition 4  The **determinant** of the matrix

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \]

is defined to be

\[ \text{det} (A) = a_{11}a_{22} - a_{12}a_{21}. \]

(An alternate notation for \( \text{det} (A) \) is \( |A| \).)

Having made the necessary conventions and definitions, we now state the main theorem regarding solutions sets of linear systems.

**Theorem 5** Given any matrix

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \]

the following statements are equivalent (meaning that either all of the statements are true or all of them are false):

1. \( \text{det} (A) \neq 0 \)
2. The homogeneous matrix equation \( A\mathbf{x} = \mathbf{0} \) has only the trivial solution.
3. There exists at least one vector \( \mathbf{b} \) such that the matrix equation \( A\mathbf{x} = \mathbf{b} \) has a unique solution.
4. For any given vector \( \mathbf{b} \), the matrix equation \( A\mathbf{x} = \mathbf{b} \) has a unique solution.
5. The set of vectors \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) (as defined above) is a linearly independent set of vectors.

**Example 6** Note that

\[ \text{det} \left( \begin{bmatrix} 2 & -5 \\ 3 & 4 \end{bmatrix} \right) = (2)(4) - (-5)(3) = 23 \neq 0. \]

We can thus conclude that the homogeneous system

\[
\begin{align*}
2x - 5y &= 0 \\
3x + 4y &= 0
\end{align*}
\]
has only the trivial solution and that the system

\[
\begin{align*}
2x - 5y &= b_1 \\
3x + 4y &= b_2
\end{align*}
\]

has a unique solution no matter what the numbers \( b_1 \) and \( b_2 \) are.

**Example 7** Note that

\[
\det \left( \begin{bmatrix} 4 & -3 \\ -8 & 6 \end{bmatrix} \right) = (4)(6) - (-3)(-8) = 0.
\]

We can thus conclude that the homogeneous system

\[
\begin{align*}
4x - 3y &= 0 \\
-8x + 6y &= 0
\end{align*}
\]

has non-trivial solutions and that, no matter how we choose the numbers \( b_1 \) and \( b_2 \), the system

\[
\begin{align*}
4x - 3y &= b_1 \\
-8x + 6y &= b_2
\end{align*}
\]

will have either infinitely many solutions or no solution.

**Exercise 8** Determine whether each of the following systems has a unique solution, infinitely many solutions, or no solution.

1. 

\[
\begin{align*}
-2x &= 0 \\
4x + 3y &= 0
\end{align*}
\]

2. 

\[
\begin{align*}
-2x &= 12 \\
4x + 3y &= 13
\end{align*}
\]

3. 

\[
\begin{align*}
x + y &= 0 \\
5x + 5y &= 0
\end{align*}
\]
4. 
\[
\begin{align*}
  x + y &= 12 \\
  5x + 5y &= 60
\end{align*}
\]

5. 
\[
\begin{align*}
  x + y &= 12 \\
  5x + 5y &= 59
\end{align*}
\]

6. Can we choose numbers \(b_1\) and \(b_2\) such that system
\[
\begin{align*}
  -2x &= b_1 \\
  4x + 3y &= b_2
\end{align*}
\]
has infinitely many solutions? If so, then give an example of such a \(b_1\) and \(b_2\).

7. Can we choose numbers \(b_1\) and \(b_2\) such that system
\[
\begin{align*}
  -2x &= b_1 \\
  4x + 3y &= b_2
\end{align*}
\]
has no solutions? If so, then give an example of such a \(b_1\) and \(b_2\).

8. Can we choose numbers \(b_1\) and \(b_2\) such that system
\[
\begin{align*}
  x + y &= b_1 \\
  5x + 5y &= b_2
\end{align*}
\]
has a unique solution? If so, then give an example of such a \(b_1\) and \(b_2\).

2 Eigenvalues and Eigenvectors of a Matrix

We will now define the concepts of “eigenvalue” and “eigenvector” for a \(2 \times 2\) matrix. These concepts are relevant for any square \((n \times n)\) matrix, but we will concentrate only on the \(2 \times 2\) case, since this is the case that will be relevant to us in this differential equations course.
Definition 9 If $A$ is a $2 \times 2$ matrix, $v$ is a non-zero vector in $\mathbb{C}^2$ (where $\mathbb{C}^2$ is the set of all vectors with two real or imaginary components), and $\lambda$ is a scalar (meaning a real or imaginary number) such that $Av = \lambda v$, then $v$ is said to be an \textit{eigenvector} of the matrix $A$ and $\lambda$ is said to be an \textit{eigenvalue} of the matrix $A$. More specifically, $v$ is said to be an eigenvector of $A$ associated with the eigenvalue $\lambda$.

Remark 10 If $A$ is any $2 \times 2$ matrix and $\lambda$ is any scalar, then it is obviously true that $A0 = \lambda0$. However, the zero vector is \textbf{not} an eigenvector of $A$. Part of the definition of “eigenvector” is that an eigenvector must be a non-zero vector.

Example 11 For the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

find the eigenvalues of $A$ and the eigenvectors associated with these eigenvalues.

Solution 12 If

$$v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an eigenvector of $A$ associated with some eigenvalue $\lambda$, then it must be the case that $v \neq 0$ and that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We can write the above equation as

$$-x_2 = \lambda x_1$$
$$x_1 = \lambda x_2$$

or as

$$-\lambda x_1 - x_2 = 0$$
$$x_1 - \lambda x_2 = 0$$
or as

\[
\begin{bmatrix}
-\lambda & -1 \\
1 & -\lambda
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]  

(11)

Our assumption that \( \mathbf{v} \neq \mathbf{0} \) means that the above homogeneous system has a non-trivial solution. This means that the coefficient matrix of the above system must have determinant zero:

\[
\det\left(\begin{bmatrix}
-\lambda & -1 \\
1 & -\lambda
\end{bmatrix}\right) = 0.
\]

Thus

\[
\lambda^2 + 1 = 0.
\]

The above equation is called the **characteristic equation** of the matrix \( \mathbf{A} \).

The solutions of this equation are

\[
\lambda = \pm i.
\]

Thus, \( \mathbf{A} \) has two eigenvalues and they are both imaginary numbers. To find the eigenvectors of \( \mathbf{A} \) associated with the eigenvalue \( \lambda = i \), we substitute \( \lambda = i \) into equation (11) to obtain

\[
\begin{bmatrix}
-i & -1 \\
1 & -i
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

which can be written in standard form as

\[
-x_1 - x_2 = 0
\]

\[
x_1 - ix_2 = 0.
\]

When we multiply the second of these equations on both sides by \(-i\), we obtain the first equation. Thus any solution of either of the equations in the system is actually a solution to both equations in the system. Solving

\[
-x_1 - x_2 = 0,
\]

we obtain

\[
x_2 = -ix_1.
\]

If we let \( x_1 \) be any scalar and then let \( x_2 = -ix_1 \), then the vector

\[
\mathbf{v} = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

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is an eigenvector of the matrix $A$ associated with the eigenvalue $\lambda = i$. A specific example of an eigenvector of $A$ associated with the eigenvalue $\lambda = i$ is

$$v = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

and there are infinitely many others (since we can choose $x_1$ however we like). To check that the above $v$ is an eigenvector (associated with $\lambda = i$), observe that

$$Av = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix} = iv = \lambda v.$$ 

As an exercise, we leave it to the reader to find the eigenvectors of $A$ associated with eigenvalue $\lambda = -i$.

**Example 13** For the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

find the eigenvalues of $A$ and the eigenvectors associated with these eigenvalues.

**Solution 14** If

$$v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an eigenvector of $A$ associated with some eigenvalue $\lambda$, then it must be the case that $v \neq 0$ and that

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$ 

We can write the above equation as

$$-x_1 = \lambda x_1$$

$$x_2 = \lambda x_2$$

or as

$$(-1 - \lambda) x_1 = 0$$

$$(1 - \lambda) x_2 = 0$$
or as
\[
\begin{bmatrix}
-1 - \lambda & 0 \\
0 & 1 - \lambda
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]
Our assumption that \( \mathbf{v} \neq \mathbf{0} \) means that the above homogeneous system has a non-trivial solution. This means that the coefficient matrix of the above system must have determinant zero:
\[
\det\left(\begin{bmatrix}
-1 - \lambda & 0 \\
0 & 1 - \lambda
\end{bmatrix}\right) = 0.
\]
Since
\[
\det\left(\begin{bmatrix}
-1 - \lambda & 0 \\
0 & 1 - \lambda
\end{bmatrix}\right) = (-1 - \lambda)(1 - \lambda),
\]
we see that if \( \lambda \) is an eigenvalue of \( A \), then
\[
(1 - \lambda)(1 - \lambda) = 0,
\]
which means that either \( \lambda = -1 \) or \( \lambda = 1 \). Both of these numbers are eigenvalues of the matrix \( A \).

To find the eigenvectors of \( A \) associated with the eigenvalue \( \lambda = -1 \), we must find the non-trivial solutions of the system
\[
\begin{bmatrix}
-1 - (-1) & 0 \\
0 & 1 - (-1)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
which is the same as
\[
\begin{bmatrix}
0 & 0 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]
The above system is clearly equivalent to
\[
2x_2 = 0
\]
and we see that any vector of the form
\[
\mathbf{v} = \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
t \\
0
\end{bmatrix} = t \begin{bmatrix}
1 \\
0
\end{bmatrix} \quad \text{(where } t \text{ can be any scalar)}
\]
is an eigenvector of \( A \) associated with the eigenvalue \( \lambda = -1 \).
To find the eigenvectors of $A$ associated with the eigenvalue $\lambda = 1$, we must find the non-trivial solutions of the system

\[
\begin{bmatrix}
-1 & -(1) & 0 \\
0 & 1 & -(1)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

which is the same as

\[
\begin{bmatrix}
-2 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

The above system is clearly equivalent to

\[-2x_1 = 0\]

and we see that any vector of the form

\[
v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

is an eigenvector of $A$ associated with the eigenvalue $\lambda = 1$.

Before proceeding to develop a general procedure for finding eigenvalues and eigenvectors, we look at two more examples:

**Example 15** Find the eigenvalues and associated eigenvectors of the matrix

\[
A = \begin{bmatrix}
2 & 0 \\
15 & -3
\end{bmatrix}
\]

**Solution 16** If

\[
v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

is an eigenvector of $A$ associated with some eigenvalue $\lambda$, then it must be the case that $v \neq 0$ and that

\[
\begin{bmatrix}
2 & 0 \\
15 & -3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \lambda \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

We can write the above equation as

\[
2x_1 = \lambda x_1 \\
15x_1 - 3x_2 = \lambda x_2
\]
or as

\[(2 - \lambda) x_1 = 0\]
\[15x_1 + (-3 - \lambda) x_2 = 0\]

or as

\[
\begin{bmatrix}
  2 - \lambda & 0 \\
  15 & -3 - \lambda
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\]

Our assumption that \( \mathbf{v} \neq \mathbf{0} \) means that the above homogeneous system has a non-trivial solution. This means that the coefficient matrix of the above system must have determinant zero:

\[
\det\left( \begin{bmatrix}
  2 - \lambda & 0 \\
  15 & -3 - \lambda
\end{bmatrix} \right) = 0.
\]

Since

\[
\det\left( \begin{bmatrix}
  2 - \lambda & 0 \\
  15 & -3 - \lambda
\end{bmatrix} \right) = (2 - \lambda)(-3 - \lambda),
\]

we see that if \( \lambda \) is an eigenvalue of \( A \), then

\[(2 - \lambda)(-3 - \lambda) = 0,
\]

which means that either \( \lambda = 2 \) or \( \lambda = -3 \). Both of these numbers are eigenvalues of the matrix \( A \).

To find the eigenvectors of \( A \) associated with the eigenvalue \( \lambda = 2 \), we must find the non-trivial solutions of the system

\[
\begin{bmatrix}
  2 - (2) & 0 \\
  15 & -3 - (2)
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

which is the same as

\[
\begin{bmatrix}
  0 & 0 \\
  15 & -5
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\]

or as

\[15x_1 - 5x_2 = 0.
\]

We can now see that any vector of the form

\[
\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 3t \end{bmatrix} = t \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\]

is an eigenvector of \( A \) associated with the eigenvalue \( \lambda = 2 \).
is an eigenvector of $A$ associated with the eigenvalue $\lambda = 2$.

To find the eigenvectors of $A$ associated with the eigenvalue $\lambda = -3$, we must find the non-trivial solutions of the system

$$
\begin{bmatrix}
2 - (-3) & 0 \\
15 & -3 - (-3)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

which is the same as

$$
\begin{bmatrix}
5 & 0 \\
15 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}.
$$

The above system is clearly equivalent to

$$
x_1 = 0
$$

and we see that any vector of the form

$$
v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

is an eigenvector of $A$ associated with the eigenvalue $\lambda = -3$.

Example 17 Find the eigenvalues and associated eigenvectors of the matrix

$$
A = \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}.
$$

Solution 18 It is easily seen that if $v$ is any vector in $\mathbb{C}^2$, then

$$
Av = 2v.
$$

Therefore, the only eigenvalue of $A$ is $\lambda = 2$ and every vector in $\mathbb{C}^2$ is an eigenvector of $A$ associated with the eigenvalue $\lambda = 2$. 

14
2.1 A General Procedure for Finding Eigenvalues and Eigenvectors

Guided by the examples that we have studied, we now develop a general procedure for finding eigenvalues and eigenvectors.

To find the eigenvalues of a $2 \times 2$ matrix, $A$, and the eigenvectors associated with these eigenvalues, we must study the equation

$$A v = \lambda v.$$ 

This equation has two unknowns: $\lambda$ (which is a scalar) and $v$ (which is a vector). Of course, one obvious solution of the above equation is $\lambda = \text{any scalar}$ and $v = 0$. However, we are interested only in identifying cases that there exist solutions for which $v \neq 0$. Such solutions, if they exist at all, will only exist for certain values of $\lambda$ (which are what we call the eigenvalues).

The first step in our solution process is to write the above equation as

$$A v = \lambda (I v)$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(the identity matrix)

and to observe that this is the same as

$$A v = (\lambda I) v$$

which is the same as

$$A v - (\lambda I) v = 0$$

which is the same as

$$(A - \lambda I) v = 0.$$ 

The above equation is a homogeneous equation with coefficient matrix $A - \lambda I$. We are seeking non-trivial solutions, $v$, of this homogeneous system. We know that the above system will have non-trivial solutions if and only if its coefficient matrix, $A - \lambda I$, has determinant zero:

$$\det(A - \lambda I) = 0.$$ 

We conclude that the eigenvalues, $\lambda$, of the matrix $A$ are precisely those values of $\lambda$ that satisfy the above equation, which is called the characteristic equation of the matrix $A$. Once we have found the eigenvalues
of \( A \), we must then proceed (for each eigenvalue \( \lambda \)) to solve the equation \((A - \lambda I)v = 0\). The solution set of this equation (which will definitely contain non–trivial solutions) is the set of all eigenvectors of \( A \) associated with the eigenvalue \( \lambda \).

**Example 19** Find the eigenvalues and associated eigenvalues of the matrix

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.
\]

**Solution 20** Since

\[
A - \lambda I = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix},
\]

the eigenvalues of \( A \) are the solutions of

\[
\det \left( \begin{bmatrix} 1 - \lambda & 2 \\ 3 & 4 - \lambda \end{bmatrix} \right) = 0
\]

which can be written as

\[
(1 - \lambda)(4 - \lambda) - 6 = 0
\]

or, in expanded form as

\[
\lambda^2 - 5\lambda - 2 = 0.
\]

Using the quadratic formula, we obtain the solutions

\[
\lambda = \frac{5 \pm \sqrt{25 - 4(-2)}}{2} = \frac{5 \pm \sqrt{33}}{2}.
\]

Therefore, the eigenvalues of \( A \) are

\[
\lambda = \frac{5 - \sqrt{33}}{2} \approx -0.372
\]

and

\[
\lambda = \frac{5 + \sqrt{33}}{2} \approx 5.372.
\]
To find the eigenvectors of $A$ associated with the eigenvalue $\lambda = \frac{5-\sqrt{33}}{2}$, we must solve the system

$$
\begin{bmatrix}
1 - \left( \frac{5-\sqrt{33}}{2} \right) & 2 \\
3 & 4 - \left( \frac{5-\sqrt{33}}{2} \right)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}.
$$

All solutions of the above homogeneous system have the form

$$
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
t
\begin{bmatrix}
3 + \sqrt{33} \\
-6
\end{bmatrix}.
$$

To find the eigenvectors of $A$ associated with the eigenvalue $\lambda = \frac{5+\sqrt{33}}{2}$, we must solve the system

$$
\begin{bmatrix}
1 - \left( \frac{5+\sqrt{33}}{2} \right) & 2 \\
3 & 4 - \left( \frac{5+\sqrt{33}}{2} \right)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}.
$$

All solutions of the above homogeneous system have the form

$$
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
t
\begin{bmatrix}
3 - \sqrt{33} \\
-6
\end{bmatrix}.
$$

Exercise 21 The purpose of this exercise is to perform a “check” of the results in the above example: Let $A$ be the matrix

$$
A = 
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}.
$$

Show that

$$
v = 
\begin{bmatrix}
3 + \sqrt{33} \\
-6
\end{bmatrix}
$$

is an eigenvector of $A$ associated with the eigenvalue

$$
\lambda = \frac{5-\sqrt{33}}{2}.
$$

Also, show that

$$
v = 
\begin{bmatrix}
3 - \sqrt{33} \\
-6
\end{bmatrix}
$$

is an eigenvector of $A$ associated with the eigenvalue

$$
\lambda = \frac{5+\sqrt{33}}{2}.
$$
Exercise 22 Find the eigenvalues and corresponding eigenvectors for the matrices, $A$, given below.

1. 
   \[ A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \]

2. 
   \[ A = \begin{bmatrix} -2 & 0 \\ 4 & 3 \end{bmatrix} \]

3. 
   \[ A = \begin{bmatrix} -2 & 2 \\ 4 & 3 \end{bmatrix} \]

4. 
   \[ A = \begin{bmatrix} -2 & 0 \\ 4 & 0 \end{bmatrix} \]

5. 
   \[ A = \begin{bmatrix} 3 & 3 \\ -1 & 3 \end{bmatrix} . \]