1. We want to find the general solution of 
\[ \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 2y = \cos(t). \]

The corresponding unforced equation 
\[ \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 2y = 0 \]
has characteristic equation 
\[ \lambda^2 + 3\lambda + 2 = 0 \]
and eigenvalues \( \lambda = -2 \) and \(-1\). The general solution of the unforced equation is thus 
\[ N(t) = c_1e^{-2t} + c_2e^{-t}. \]

The complexification of the forced equation is 
\[ \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 2y = e^{it} \]
and our guess at a particular solution of the complexified equation is 
\[ y = ae^{it} \]
\[ \frac{dy}{dt} = aie^{it} \]
\[ \frac{d^2y}{dt^2} = -ae^{it}. \]

If this guess is to be correct, then we must have 
\[ (-ae^{it}) + 3(iae^{it}) + 2(ae^{it}) = e^{it} \]
or equivalently 
\[ ae^{it}(1 + 3i) = e^{it}. \]

This gives us 
\[ a = \frac{1}{1 + 3i} = \frac{1}{1 + 3i} \cdot \frac{1 - 3i}{1 - 3i} = \frac{1 - 3i}{10} = \frac{1}{10} - \frac{3}{10}i. \]

We can now write our guess, \( y = ae^{it} \), as 
\[ y = \left( \frac{1}{10} - \frac{3}{10}i \right)(\cos(t) + i\sin(t)) \]
\[ = \left( \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t) \right) + i\left( -\frac{3}{10} \cos(t) + \frac{1}{10} \sin(t) \right). \]
The real part of this solution of the complexified equation should be a solution of the original equation. Thus, we claim that a particular solution of
\[
\frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 2y = \cos(t)
\]
is
\[
F(t) = \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t).
\]
Let us check that this correct: Using the \(F\) that we claim is a solution, we have
\[
\frac{dF}{dt} = -\frac{1}{10} \sin(t) + \frac{3}{10} \cos(t)
\]
\[
\frac{d^2F}{dt^2} = -\frac{1}{10} \cos(t) - \frac{3}{10} \sin(t).
\]
We observe that
\[
\frac{d^2F}{dt^2} + 3 \frac{dF}{dt} + 2y = \left( -\frac{1}{10} \cos(t) - \frac{3}{10} \sin(t) \right) + 3 \left( -\frac{1}{10} \sin(t) + \frac{3}{10} \cos(t) \right) + 2 \left( \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t) \right)
\]
\[
= \cos(t)
\]
showing that \(F\) is indeed a solution.

The general solution of the original forced equation is
\[
y = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t).
\]
The forced response oscillates with period \(2\pi\) and amplitude
\[
|a| = \sqrt{\left( -\frac{1}{10} \right)^2 + \left( -\frac{3}{10} \right)^2} = \sqrt{\frac{1}{10}} \approx 0.316.
\]
If we pull the mass far from its rest position and let go, the mass will move slowly back toward its rest position (slowly because we are in the overdamped case) but once it gets close to the rest position, it will oscillate about the rest position (due to the forcing) with period approximately equal to \(2\pi\) and amplitude approximately equal to 0.316.

Here is a specific example: Suppose we find the solution of the forced equation that satisfies the initial conditions \(y(0) = 5\) and \(y'(0) = 0\). Using the general solution
\[
y = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t)
\]
which has derivative
\[
y' = -2c_1 e^{-2t} - c_2 e^{-t} - \frac{1}{10} \sin(t) + \frac{3}{10} \cos(t)
\]
and using the given initial conditions, we obtain
\[
c_1 + c_2 + \frac{1}{10} = 5
\]
\[
-2c_1 - c_2 + \frac{3}{10} = 0
\]
and thus
\[
c_1 = -\frac{23}{5},
\]
\[
c_2 = \frac{19}{2}.
\]
The particular solution satisfying \(y(0) = 5, y'(0) = 0\) is thus
\[
y = -\frac{23}{5}e^{-2t} + \frac{19}{2}e^{-t} + \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t).
\]
A graph of this solution is shown below.

15. Instead of using the equation given in the book, let us use the same one as used in problem 1:
\[
\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \cos(t).
\]
We know (from problem 1) that a particular solution of this forced equation is
\[
F(t) = \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t).
\]
The point of this exercise is to study two other ways to go about finding this particular solution (without using complexification as we did in problem 1).

a. We guess that the given equation has a solution of the form
\[
y = a\cos(t) + b\sin(t).
\]
The problem is then to find the correct values of \(a\) and \(b\).
For our guess, we have
\[
y' = -a\sin(t) + b\cos(t)
\]
\[
y'' = -a\cos(t) - b\sin(t).
\]
Substituting this guess into the differential equation gives us
\((-a \cos(t) - b \sin(t))\)  
\(+ 3(-a \sin(t) + b \cos(t))\)  
\(+ 2(a \cos(t) + b \sin(t))\)  
\(= \cos(t)\)

which is the same as
\((a + 3b) \cos(t) + (-3a + b) \sin(t) = \cos(t)\).

Since we want this equation to be true for all real numbers \(t\), we can set \(t = 0\) to obtain
\[a + 3b = 1\]
and set \(t = \pi/2\) to obtain
\[-3a + b = 0.\]

The solution of the above system of equations is
\[a = \frac{1}{10}\]
\[b = \frac{3}{10}.\]

b. We guess that we have a solution of the form
\[y = A \cos(t - \phi).\]

The problem is to determine the correct choices of \(A\) and \(\phi\).

Since, for our guess, we have
\[
\frac{dy}{dt} = -A \sin(t - \phi) \\
\frac{d^2 y}{dt^2} = -A \cos(t - \phi),
\]
we must have
\[-A \cos(t - \phi) - 3A \sin(t - \phi) + 2A \cos(t - \phi) = \cos(t)\]
which can be written as
\[A \cos(t - \phi) - 3A \sin(t - \phi) = \cos(t).\]

We want the above equation to be true for all real numbers \(t\).

Choosing \(t = \phi\), we obtain
\[A = \cos(\phi)\]
and choosing \(t = \pi/2 + \phi\), we obtain
\[-3A = \cos\left(\frac{\pi}{2} + \phi\right).\]

By a familiar trigonometric identity, we have
\[\cos\left(\frac{\pi}{2} + \phi\right) = -\sin(\phi).\]

Therefore, our choices for \(A\) and \(\phi\) must satisfy both of the equations.
\[ A = \cos(\phi) \]
\[ 3A = \sin(\phi). \]

Squaring both sides of the above equations and then adding the results gives
\[ A^2 + 9A^2 = 1 \]
or
\[ A = \sqrt{\frac{1}{10}}. \]

We have now found \( A \). To find \( \phi \), we note that we must have
\[ \tan(\phi) = 3 \]
so we can take
\[ \phi = \arctan(3) \approx 1.25. \]

A solution of our forced equation is thus
\[ y = \sqrt{\frac{1}{10}} \cos(t - \arctan(3)). \]

Let us use some trigonometric identities to check to see that this is actually the same solution that was found in problem 1 and in part a: Note that
\[ y = \sqrt{\frac{1}{10}} \cos(t - \arctan(3)) \]
\[ = \sqrt{\frac{1}{10}} [\cos(t) \cos(\arctan(3)) + \sin(t) \sin(\arctan(3))] \]

Using the fact that for any real number, \( x \), we have
\[ \cos(\arctan(x)) = \frac{1}{\sqrt{1 + x^2}} \]
\[ \sin(\arctan(x)) = \frac{x}{\sqrt{1 + x^2}}, \]
we obtain
\[ y = \sqrt{\frac{1}{10}} \left[ \cos(t) \left( \frac{1}{\sqrt{1 + 3^2}} \right) + \sin(t) \left( \frac{3}{\sqrt{1 + 3^2}} \right) \right] \]
\[ = \sqrt{\frac{1}{10}} \left[ \frac{1}{\sqrt{10}} \cos(t) + \frac{3}{\sqrt{10}} \sin(t) \right] \]
\[ = \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t). \]

21. Let us change the problem to
\[ \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 2y = 5 \cos(t). \]

As seen in problem 1, the unforced equation has general solution
\[ N(t) = c_1 e^{-2t} + c_2 e^{-t}. \]
By the result in problem 1 and the result of problem 20, the forced equation has solution
\[ F(t) = 5 \left( \frac{1}{10} \cos(t) + \frac{3}{10} \sin(t) \right) = \frac{1}{2} \cos(t) + \frac{3}{2} \sin(t). \]
The general solution of the above forced equation is thus
\[ y = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{2} \cos(t) + \frac{3}{2} \sin(t). \]