Answers to Section 5.1 Homework Exercises

1. The linearization of system (i) near (0, 0) is the system
\[
\frac{dx}{dt} = 2x + y
\]
\[
\frac{dy}{dt} = -y.
\]
This linear system has coefficient matrix
\[
A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.
\]
The matrix \(A\) has eigenvalues 2 and \(-1\), which means that (0, 0) is a saddle for the linearized system and hence a local saddle for the original nonlinear system.

The linearization of system (ii) near (0, 0) is the system
\[
\frac{dx}{dt} = 2x + y
\]
\[
\frac{dy}{dt} = y.
\]
This linear system has coefficient matrix
\[
A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.
\]
The matrix \(A\) has eigenvalues 2 and 1, which means that (0, 0) is a source for the linearized system and hence a local source for the original nonlinear system.

The linearization of system (iii) near (0, 0) is the system
\[
\frac{dx}{dt} = 2x + y
\]
\[
\frac{dy}{dt} = -y.
\]
This linear system has coefficient matrix
\[
A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.
\]
The matrix \(A\) has eigenvalues 2 and \(-1\), which means that (0, 0) is a saddle for the linearized system and hence a local saddle for the original nonlinear system.

In conclusion, nonlinear systems (i) and (iii) have phase planes with the same “local picture” near (0, 0).

2. In this exercise, we use the facts that
\[
\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \text{ for all } x
\]
and
\[
\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots \text{ for all } x.
\]
(These are the Maclaurin series expansions of the sine and cosine functions.)

In particular, if \(x\) is very small (near zero), then \(\sin x = x\) and \(\cos x = 1\).

The linearization of system (i) near (0, 0) is the system
\[
\frac{du}{dt} = 3x + y \\
\frac{dv}{dt} = 4x.
\]

This linear system has coefficient matrix
\[
A = \begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix}.
\]

The matrix \(A\) has eigenvalues 4 and −1, which means that \((0, 0)\) is a saddle for the linearized system and hence a local saddle for the original nonlinear system.

The linearization of system (ii) near \((0, 0)\) is the system
\[
\frac{dx}{dt} = -3x + y \\
\frac{dy}{dt} = 4x.
\]

This linear system has coefficient matrix
\[
A = \begin{pmatrix} -3 & 1 \\ 4 & 0 \end{pmatrix}.
\]

The matrix \(A\) has eigenvalues −4 and 1, which means that \((0, 0)\) is a saddle for the linearized system and hence a local saddle for the original nonlinear system.

The linearization of system (iii) near \((0, 0)\) is the system
\[
\frac{dx}{dt} = -3x + y \\
\frac{dy}{dt} = 4x.
\]

This linear system has coefficient matrix
\[
A = \begin{pmatrix} -3 & 1 \\ 4 & 0 \end{pmatrix}.
\]

The matrix \(A\) has eigenvalues −4 and 1, which means that \((0, 0)\) is a saddle for the linearized system and hence a local saddle for the original nonlinear system.

All three nonlinear systems have local saddles at \((0, 0)\). Systems (ii) and (iii), however, are the most similar to each other because they have the same eigenvalues (and hence the eigenvectors of their linearizations are the same).

3. Consider the system
\[
\frac{dx}{dt} = -2x + y \\
\frac{dy}{dt} = -y + x^2.
\]

a. The linearized system at \((0, 0)\) is
\[
\frac{dx}{dt} = -2x + y \\
\frac{dy}{dt} = -y.
\]

b. The coefficient matrix of the linearized system is
\[
A = \begin{pmatrix}
-2 & 1 \\
0 & -1
\end{pmatrix}.
\]

Since \(A\) has eigenvalues \(-2\) and \(-1\), \((0,0)\) is a sink.

c. Phase portrait for the nonlinear system near \((0,0)\):

\[
\frac{du}{dt} = \frac{dx}{dt} = -2x + y = -2(u + 2) + (v + 4) = -2u - 4 + v + 4 = -2u + v
\]

and

\[
\frac{dv}{dt} = \frac{dy}{dt} = -y + x^2 = -(v + 4) + (u + 2)^2 = -v - 4 + u^2 + 4u + 4 = 4u - v + u^2.
\]

The (nonlinear) system
\[ \begin{align*}
\frac{du}{dt} &= -2u + v \\
\frac{dv}{dt} &= 4u - v + u^2
\end{align*} \]

has an equilibrium point at \((0, 0)\) as expected. The linearization of this system at \((0, 0)\) (which is also the linearization of the original nonlinear system at \((2, 4)\)) is:

\[ \begin{align*}
\frac{du}{dt} &= -2u + v \\
\frac{dv}{dt} &= 4u - v.
\end{align*} \]

Since the coefficient matrix,

\[ A = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix} \]

has eigenvalues \(-\frac{3}{2} + \frac{1}{2} \sqrt{17} > 0\) and \(-\frac{3}{2} - \frac{1}{2} \sqrt{17} < 0\), we see that \((2, 4)\) is a saddle for the original nonlinear system. The phase portrait for the original nonlinear system near \((2, 4)\) is shown below.

Here is a more global picture of the original system.
4. Consider the system

\[
\begin{align*}
\frac{dx}{dt} &= -x \\
\frac{dy}{dt} &= -4x^3 + y.
\end{align*}
\]

a. The origin is the only equilibrium point of this system because the only solution of the system of equations

\[
\begin{align*}
-x &= 0 \\
-4x^3 + y &= 0
\end{align*}
\]

is \((x, y) = (0, 0)\).

b. The linearized system at the origin is

\[
\begin{align*}
\frac{dx}{dt} &= -x \\
\frac{dy}{dt} &= y.
\end{align*}
\]

c. The equilibrium point \((0, 0)\) is a saddle for the linearized system. The phase plane (for the linearized system) is shown below.
5. We are considering the nonlinear system
\[
\frac{dx}{dt} = -x \\
\frac{dy}{dt} = -4x^3 + y.
\]

a. The general solution of the differential equation
\[
\frac{dx}{dt} = -x
\]
is
\[x = C_1 e^{-t} \text{ (where } C_1 \text{ can be any constant).}\]

b. Since \(x = C_1 e^{-t}\), we have
\[
\frac{dy}{dt} = -4(C_1 e^{-t})^3 + y
\]
which can be written as
\[
\frac{dy}{dt} - y = -4C_1^3 e^{-3t}.
\]
Using the integrating factor \(\mu = e^{-t}\), we obtain
\[e^{-t} \cdot \frac{dy}{dt} - e^{-t} \cdot y = -4C_1^3 e^{-4t}\]
which can be written as
\[
\frac{d}{dt}(e^{-t} \cdot y) = -4C_1^3 e^{-4t}.
\]
The above equation tells us that \(e^{-t} \cdot y\) must be an antiderivative of \(-4C_1^3 e^{-4t}\). Since
\[
\int -4C_1^3 e^{-4t} \, dt = C_1^3 e^{-4t} + C_2,
\]
we obtain
\[e^{-t} \cdot y = C_1^3 e^{-4t} + C_2\]
which gives us
\[ y = C_1 e^{-3t} + C_2 e^t. \]

c. The general solution of our system is
\[ x = C_1 e^{-t} \]
\[ y = C_1 e^{-3t} + C_2 e^t \]
(where \(C_1\) and \(C_2\) can be any constants).

d. No matter what \(C_1\) is, it must be the case that \(\lim_{t \to \infty} x(t) = 0\). In order for \(\lim_{t \to \infty} y(t) = 0\), we must have \(C_2 = 0\) (for otherwise, the term \(C_2 e^t\) would blow up as \(t \to \infty\)). Thus the solutions which approach the origin as \(t \to \infty\) are precisely those for which \(C_2 = 0\).

e. In order to have \(\lim_{t \to -\infty} x(t) = 0\), it must be the case that \(C_1 = 0\). If \(C_1 = 0\), then \(y = C_2 e^t\) which approaches 0 as \(t \to -\infty\) no matter what \(C_2\) is. Thus, the solutions which approach the origin as \(t \to -\infty\) are precisely those for which \(C_1 = 0\).

f. Note that if \(C_2 = 0\), we have
\[ x = C_1 e^{-t} \]
\[ y = C_1 e^{-3t} = (C_1 e^{-t})^3 \]
and hence \(y = x^3\). The curve \(y = x^3\) is a separatrix.

If \(C_1 = 0\), then we have
\[ x = 0 \]
\[ y = C_2 e^t. \]

The curve \(x = 0\) (the \(y\) axis) is thus a separatrix. A picture of the two separatrices is shown below.

g. For the linearized system in Exercise 4, the separatrices are the \(x\) and \(y\) axes. For the nonlinear system in this exercise, the separatrices are the \(x\) axis and the curve \(y = x^3\). In both cases, the \(x\) axis is a “stable”
separatrix - meaning that solutions that start on this separatrix stay on it for all time and approach the origin as $t \to \infty$. In the linear system of Exercise 4, the $y$ axis is the “unstable” separatrix - meaning that solutions that start on this separatrix stay on it for all time and approach the origin as $t \to -\infty$. The unstable separatrix for the nonlinear system is the curve $y = x^3$.

The figure below shows the phase portrait for the nonlinear system of this exercise.

7. We are considering the system

\[
\frac{dx}{dt} = x(-x - 3y + 150) \\
\frac{dy}{dt} = y(-2x - y + 100).
\]

a. To find the equilibrium points of this system, we must solve the system of equations

\[
x(-x - 3y + 150) = 0 \\
y(-2x - y + 100) = 0.
\]

The solutions of this system are $(0, 0), (0, 100), (150, 0),$ and $(30, 40)$. All of these are in the first quadrant.

To classify the equilibria, it will convenient to compute the Jacobian matrix for the pair of functions

\[
f(x, y) = -x^2 - 3xy + 150x
\]

and

\[
g(x, y) = -2xy - y^2 + 100y.
\]

We have
\[ \frac{\partial f}{\partial x} = -2x - 3y + 150 \]
\[ \frac{\partial f}{\partial y} = -3x \]
\[ \frac{\partial g}{\partial x} = -2y \]
\[ \frac{\partial g}{\partial y} = -2x - 2y + 100 \]

which gives us the Jacobian matrix
\[
J(x, y) = \begin{pmatrix}
-2x - 3y + 150 & -3x \\
-2y & -2x - 2y + 100
\end{pmatrix}.
\]

At the equilibrium point \((0, 0)\), we have
\[
J(0, 0) = \begin{pmatrix}
150 & 0 \\
0 & 100
\end{pmatrix}
\]
and this matrix has eigenvalues 100 and 150, which means that \((0, 0)\) is a source.

At the equilibrium point \((0, 100)\), we have
\[
J(0, 100) = \begin{pmatrix}
-150 & 0 \\
-200 & -100
\end{pmatrix}
\]
and this matrix has eigenvalues \(-100\) and \(-150\), which means that \((0, 100)\) is a sink.

At the equilibrium point \((150, 0)\), we have
\[
J(150, 0) = \begin{pmatrix}
-150 & -450 \\
0 & -200
\end{pmatrix}
\]
and this matrix has eigenvalues \(-150\) and \(-200\), which means that \((150, 0)\) is a sink.

Finally, at the equilibrium point \((30, 40)\), we have
\[
J(30, 40) = \begin{pmatrix}
-30 & -90 \\
-80 & -40
\end{pmatrix}
\]
and this matrix has eigenvalues \(-120\) and \(50\), which means that \((150, 0)\) is a saddle.

**b-c.** The global picture (phase portrait) for this system (in the first quadrant) is shown below.


9. We are considering the system
\[
\begin{align*}
\frac{dx}{dt} &= x(100 - x - 2y) \\
\frac{dy}{dt} &= y(150 - x - 6y).
\end{align*}
\]

a. The solutions of
\[
\begin{align*}
x(100 - x - 2y) &= 0 \\
y(150 - x - 6y) &= 0
\end{align*}
\]
are (0, 0), (100, 0), (0, 25), and (75, 12.5) and these are the equilibrium points of the system.

We have
\[
f(x, y) = 100x - x^2 - 2xy
\]
and
\[
g(x, y) = 150y - xy - 6y^2
\]
which gives us
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 100 - 2x - 2y \\
\frac{\partial f}{\partial y} &= -2x \\
\frac{\partial g}{\partial x} &= -y \\
\frac{\partial g}{\partial y} &= 150 - x - 12y
\end{align*}
\]
so the Jacobian matrix is
\[
J(x, y) = \begin{pmatrix} 100 - 2x - 2y & -2x \\
-y & 150 - x - 12y \end{pmatrix}.
\]

At the equilibrium point (0, 0), we have
\[ J(0, 0) = \begin{pmatrix} 100 & 0 \\ 0 & 150 \end{pmatrix} \]

and the eigenvalues of this matrix are 100 and 150, meaning that (0, 0) is a source.

At the equilibrium point (100, 0), we have
\[ J(100, 0) = \begin{pmatrix} -100 & -200 \\ 0 & 50 \end{pmatrix} \]

and the eigenvalues of this matrix are -100 and 50, meaning that (100, 0) is a saddle.

At the equilibrium point (0, 25), we have
\[ J(0, 25) = \begin{pmatrix} 50 & 0 \\ -25 & -150 \end{pmatrix} \]

and the eigenvalues of this matrix are -150 and 50, meaning that (0, 25) is a saddle.

Finally, at the equilibrium point (75, 12.5), we have
\[ J(75, 12.5) = \begin{pmatrix} -75 & -150 \\ -12.5 & -75 \end{pmatrix} \]

and the eigenvalues of this matrix are approximately -31.7 and -118.3 meaning that (75, 12.5) is a sink.

**b-c.** The global picture (phase portrait) for this system (in the first quadrant) is shown below.

17. Consider the system
\[
\begin{align*}
\frac{dx}{dt} &= -x^3 \\
\frac{dy}{dt} &= -y + y^2.
\end{align*}
\]

a. For \( f(x, y) = -x^3 \) and \( g(x, y) = -y + y^2 \),
we have
\[
\begin{align*}
\frac{\partial f}{\partial x} &= -3x^2 \\
\frac{\partial f}{\partial y} &= 0 \\
\frac{\partial g}{\partial x} &= 0 \\
\frac{\partial g}{\partial y} &= -1 + 2y
\end{align*}
\]
so the Jacobian matrix is
\[
J(x, y) = \begin{pmatrix}
-3x^2 & 0 \\
0 & -1 + 2y
\end{pmatrix}.
\]

At the equilibrium point \((0, 0)\), we have
\[
J(0, 0) = \begin{pmatrix}
0 & 0 \\
0 & -1
\end{pmatrix}
\]
so the linearized system at \((0, 0)\) is
\[
\begin{align*}
\frac{dx}{dt} &= 0 \\
\frac{dy}{dt} &= -y.
\end{align*}
\]

b. The eigenvalues of the linearized system at \((0, 0)\) are \(\lambda_1 = -1\) and \(\lambda_2 = 0\). To find an eigenvector for \(\lambda_1 = -1\), we solve \((A - (-1) \cdot I)v = 0\), which can be written as
\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
We see that
\[
v_1 = \begin{pmatrix}
0 \\
1
\end{pmatrix}
\]
is an eigenvector for \(\lambda_1 = -1\).
To find an eigenvector for \(\lambda_2 = 0\), we solve \((A - 0 \cdot I)v = 0\), which can be written as
\[
\begin{pmatrix}
0 & 0 \\
0 & -1
\end{pmatrix}\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
We see that

$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is an eigenvector for $\lambda_2 = 0$.

The phase plane for this linearized system is shown below.

c. At the equilibrium point $(0, 1)$, we have

$$J(0, 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

so the linearized system at $(0, 1)$ is

$$\frac{dx}{dt} = 0$$
$$\frac{dy}{dt} = y.$$  

d. The eigenvalues of the linearized system at $(0, 1)$ are $\lambda_1 = 1$ and $\lambda_2 = 0$.
To find an eigenvector for $\lambda_1 = 1$, we solve $(A - 1 \cdot I)v = 0$, which can be written as

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

We see that

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is an eigenvector for $\lambda_1 = 1$.

To find an eigenvector for $\lambda_2 = 0$, we solve $(A - 0 \cdot I)v = 0$, which can be
written as
\[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

We see that
\[
v_2 = \begin{pmatrix}
1 \\
0
\end{pmatrix}
\]
is an eigenvector for \(\lambda_2 = 0\).

The phase plane for this linearized system is shown below.

\[\text{e. For the phase portrait for the nonlinear system: the equilibrium point (0,0) is a local sink and the equilibrium point (0,1) is a local saddle.}\]

\[\text{f. Whenever the linearization has 0 as an eigenvalue (as both linearizations in this problem do), then we can't use the linearization to classify the equilibrium points of the nonlinear system under consideration.}\]