Some Integrals We Can Compute

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Abstract

This is a review on how to compute some integrals.

1 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FTC) tells us that if a function, \( f \), is continuous on the interval \([a, b]\) and the function \( F \) is any antiderivative of \( f \) on \([a, b]\), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

In order to apply this result, we need to be able to find an antiderivative, \( F \), of the integrand, \( f \). Sometimes this is impossible, but there are many cases in which it can be done. Such cases are studied in elementary calculus courses (Calculus II at KSU) usually under the heading of “integration techniques”. It is our purpose here to review some of these integration techniques, most (but perhaps not all) of which were probably included in whatever Calculus II course you took.

We will employ the idea of the “indefinite integral” of \( f \), written as

\[
\int f(x) \, dx,
\]

to mean the set of all antiderivatives of \( f \) on some interval, \( I \). If \( F \) is any particular antiderivative of \( f \) on the interval \( I \), then it is known (and was perhaps proved in Calculus I or II) that

\[
\int f(x) \, dx = F(x) + C \quad \text{(where } C \text{ can be any real constant)}.\]
Since the interval, $I$, does not appear anywhere in the notation for the indefinite integral, it is not a very good notation. This usually causes no problems, however, in a course such as this where we mostly consider functions whose antiderivatives do not depend on which interval in the domain of $f$ we are considering. For example, there is no ambiguity when we write

$$\int x^2 \, dx = \frac{1}{3} x^3 + C$$

or

$$\int \cos(x) \, dx = \sin(x) + C$$

because the corresponding differentiation formulas

$$\frac{d}{dx} \left( \frac{1}{3} x^3 + C \right) = x^2$$

and

$$\frac{d}{dx} (\sin(x) + C) = \cos(x)$$

are correct on any subinterval of $(-\infty, \infty)$. One notable exception (that we do encounter very often in this course) is the function $f(x) = 1/x$. In particular, the statement

$$\int \frac{1}{x} \, dx = \ln(x) + C$$

is correct if we are considering the function $f(x) = 1/x$ with domain a subinterval of $(0, \infty)$, but it is not correct if the domain under consideration is a subinterval $(-\infty, 0)$. In the latter case, a correct statement is

$$\int \frac{1}{x} \, dx = \ln(-x) + C.$$  

The usual way in which most textbooks attempt to remove the possibility of confusion is to write

$$\int \frac{1}{x} \, dx = \ln|x| + C.$$  

This last statement is correct in both the case that the domain of $f(x) = 1/x$ is a subinterval $(0, \infty)$ and the case that the domain of $f(x) = 1/x$ is a subinterval of $(-\infty, 0)$.  

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2 A Basic Beginning Toolbox of Integrals

Below is a list of some basic integrals. These are integrals that should be memorized. All of the integration techniques that we use to compute more complicated integrals are aimed at reducing the more complicated integrals to one of the forms in the basic list.

1. If $n$ is any fixed real number (except $-1$), then
   \[ \int x^n \, dx = \frac{1}{n+1} x^{n+1} + C. \]

2. \[ \int x^{-1} \, dx = \ln |x| + C \]

3. \[ \int e^x \, dx = e^x + C \]

4. \[ \int \cos(x) \, dx = \sin(x) + C \]

5. \[ \int \sin(x) \, dx = -\cos(x) + C \]

6. \[ \int \sec^2(x) \, dx = \tan(x) + C \]

7. \[ \int csc^2(x) \, dx = -\cot(x) + C \]

8. \[ \int \sec(x) \tan(x) \, dx = \sec(x) + C \]

9. \[ \int csc(x) \cot(x) \, dx = -csc(x) + C \]
10. \[ \int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin(x) + C \]

11. \[ \int \frac{1}{1+x^2} \, dx = \arctan(x) + C. \]

The last two integrals in this basic list involve inverse trigonometric functions. Since these two are usually the least familiar to students who have completed Calculus, we will give a derivation of formula number 11.

First, we recall the definition of the inverse tangent function, which we denote by “arctan”. (Another notation for the inverse tangent function is “\(\tan^{-1}\)”.) The statement

\[ y = \arctan(x) \]

means that

\[ \tan(y) = x \]

and that

\[ -\frac{\pi}{2} < y < \frac{\pi}{2}. \]

(Note that for any given real number, \(x\), there is exactly one real number \(y\) in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\) such that \(\tan(y) = x\). This is what makes the inverse tangent function be a well-defined function.)

To find the derivative with respect to \(x\) of the function

\[ y = \arctan(x), \]

we use implicit differentiation. In particular, we begin with the equivalent formula

\[ \tan(y) = x \]

and differentiate both sides of this formula with respect to \(x\) to obtain

\[ \frac{d}{dx} (\tan(y)) = \frac{d}{dx} (x). \]

Recalling that \(y\) is a function of \(x\) (and hence that the Chain Rule must be used in differentiating the left hand side of the above equation), we obtain

\[ \sec^2(y) \frac{dy}{dx} = 1 \]
or
\[
\frac{dy}{dx} = \frac{1}{\sec^2(y)}.
\]
By a familiar trigonometric identity, we now have
\[
\frac{dy}{dx} = \frac{1}{1 + \tan^2(y)},
\]
and since \(\tan(y) = x\), we can write the above formula as
\[
\frac{dy}{dx} = \frac{1}{1 + x^2}.
\]
This proves that
\[
\frac{d}{dx}(\arctan(x)) = \frac{1}{1 + x^2}
\]
and also proves the integration formula 11 in the above list of basic integrals.

**Exercise 1** Prove integration formula number 10 in the above list. To do this, you will need to recall the definition of the inverse sine function: When we say that
\[
y = \arcsin(x),
\]
we mean that
\[
\sin(y) = x
\]
and that
\[
-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.
\]

### 3 Integration by Substitution

The Chain Rule tells us that if \(F\) and \(g\) are differentiable functions, then
\[
\frac{d}{dx}(F(g(x))) = F'(g(x)) \cdot g'(x).
\]
Thus,
\[
\int F'(g(x)) \cdot g'(x) \, dx = F(g(x)) + C.
\]
If we make the substitutions
\[
u = g(x)
\]
and

\[ du = g'(x) \, dx, \]

then we can write the above integration formula as

\[ \int F'(u) \, du = F(u) + C \]

This leads to the idea of “integration by substitution”.

**Example 2** Evaluate the indefinite integral

\[ \int x^2 (4x^3 + 5)^8 \, dx. \]

**Solution 3** Make the substitution

\[
\begin{align*}
    u & = 4x^3 + 5 \\
    du & = 12x^2 \, dx
\end{align*}
\]

and note that

\[ x^2 \, dx = \frac{1}{12} \, du. \]

The above integral can then be written as

\[ \int \frac{1}{12} u^8 \, du. \]

Since

\[ \int \frac{1}{12} u^8 \, du = \frac{1}{12} \cdot \frac{1}{9} u^9 + C, \]

we see that

\[ \int x^2 (4x^3 + 5)^8 \, dx = \frac{1}{108} (4x^3 + 5)^9 + C. \]

**Example 4** Evaluate the definite integral

\[ \int_{0}^{\pi/4} \tan^2(x) \sec^2(x) \, dx. \]
Solution 5 Making the substitution

\[ u = \tan(x) \]
\[ du = \sec^2(x) \, dx, \]

we see that

\[ \int \tan^2(x) \sec^2(x) \, dx = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} \tan^3(x) + C. \]

By the FTC, we then have

\[ \int_0^{\pi/4} \tan^2(x) \sec^2(x) \, dx = \frac{1}{3} \tan^3\left(\frac{\pi}{4}\right) - \frac{1}{3} \tan^3(0) = \frac{1}{3}. \]

(Recall that \(\tan\left(\frac{\pi}{4}\right) = 1\) and \(\tan(0) = 0\).)

Exercise 6 Evaluate the following indefinite or definite integrals.

1. \[ \int \sin(3x^2 + 2x - 1) \cdot (3x + 1) \, dx \]

2. \[ \int_0^{\ln(3)} e^x \sqrt{e^x + 4} \, dx \]

3. \[ \int \frac{3x^2 + 8x - 6}{x^3 + 4x^2 - 6x - 2} \, dx \]

4. \[ \int_0^{\pi/4} \tan(x) \, dx \]

5. \[ \int \frac{e^x}{e^{2x} + 1} \, dx \]

Hint: \(e^{2x} = (e^x)^2\).
4 Integration By Parts

The Product Rule tells us that if \( f \) and \( g \) are differentiable functions, then

\[
\frac{d}{dx} (f(x) \cdot g(x)) = f(x) \cdot g'(x) + f'(x) \cdot g(x).
\]

Thus

\[
\int (f(x) \cdot g'(x) + f'(x) \cdot g(x)) \, dx = f(x) \cdot g(x) + C
\]

or

\[
\int f(x) \cdot g'(x) \, dx + \int f'(x) \cdot g(x) \, dx = f(x) \cdot g(x) + C
\]

Making the substitutions

\[
\begin{align*}
    u &= f(x) \\
    du &= f'(x) \, dx \\
    v &= g(x) \\
    dv &= g'(x) \, dx,
\end{align*}
\]

we can write the above integration formula as

\[
\int u \, dv + \int v \, du = uv + C
\]

or as

\[
\int u \, dv = uv - \int v \, du + C.
\]

(It is not really necessary to write the “+C” at the end of the above formula, since the indefinite integral on the right automatically includes a “+C”.)

The above formula is what we call the formula for integration by parts. It is used in cases where the integral \( \int v \, du \) is easy to compute, but the integral \( \int u \, dv \) is not. Specifically, it is used when we want to compute \( \int u \, dv \), and we do this via computing \( \int v \, du \).

**Example 7** Evaluate the indefinite integral

\[
\int x \cos(x) \, dx.
\]
Solution 8  The above integral has the form $\int u \, dv$ where

\[
\begin{array}{c|c}
  u = x & dv = \cos(x) \, dx \\
  du = dx & v = \sin(x)
\end{array}
\]

Using the integration by parts formula, we obtain

\[
\int x \cos(x) \, dx = \int u \, dv = \int u \, dv = uv - \int v \, du = x \sin(x) - \int \sin(x) \, dx = x \sin(x) + \cos(x) + C.
\]

We conclude that

\[
\int x \cos(x) \, dx = x \sin(x) + \cos(x) + C.
\]

Sometimes, it is necessary to use integration by parts more than once, as in the next example.

**Example 9** Evaluate the indefinite integral

\[
\int e^x \cos(x) \, dx.
\]

**Solution 10** Make the following substitutions

\[
\begin{array}{c|c}
  u = e^x & dv = \cos(x) \, dx \\
  du = e^x \, dx & v = \sin(x)
\end{array}
\]

Then

\[
\int e^x \cos(x) \, dx = \int u \, dv = uv - \int v \, du = e^x \sin(x) - \int e^x \sin(x) \, dx.
\]
We now see that
\[ \int e^x \cos(x) \, dx = e^x \sin(x) - \int e^x \sin(x) \, dx. \] (1)

At this point, it appears that integration by parts has perhaps been of no help, because \( \int v \, du \) appears to be no easier than the original problem, \( \int u \, dv \). However, we continue (using integration by parts again) with the new problem
\[ \int e^x \sin(x) \, dx. \]

Here we make the substitutions

\[
\begin{array}{c|c}
 u = e^x & dv = \sin(x) \, dx \\
 du = e^x \, dx & v = -\cos(x)
\end{array}
\]

and obtain
\[
\int e^x \sin(x) \, dx = uv - \int v \, du = -e^x \cos(x) + \int e^x \cos(x) \, dx.
\]

Thus,
\[ \int e^x \sin(x) \, dx = -e^x \cos(x) + \int e^x \cos(x) \, dx. \] (2)

Now, combining the above results (equations (1) and (2)), we observe that
\[
\int e^x \cos(x) \, dx = e^x \sin(x) - \int e^x \sin(x) \, dx
\]
\[= e^x \sin(x) - \left(-e^x \cos(x) + \int e^x \cos(x) \, dx\right)\]
\[= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) \, dx\]

or, in summary,
\[ \int e^x \cos(x) \, dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) \, dx. \] (3)
Adding $\int e^x \cos (x) \, dx$ to both sides of equation (3) gives us
\[ 2 \int e^x \cos (x) \, dx = e^x \sin (x) + e^x \cos (x). \]
We then divide both sides of the above equation by 2 to obtain
\[ \int e^x \cos (x) \, dx = \frac{1}{2} e^x (\sin (x) + \cos (x)) + C. \]

**Exercise 11** Evaluate the following integrals.

1. \[ \int x \ln (x) \, dx \]
   *Hint: Let $u = \ln (x)$ and $dv = x \, dx$."

2. \[ \int x \sin (x) \, dx \]

3. \[ \int e^x \sin (x) \, dx \]

4. \[ \int_0^1 x e^{2x} \, dx \]

5. \[ \int x \sin (4x + 2) \, dx \]

6. \[ \int_0^{\pi/2} x^2 \cos (3x) \, dx \]

7. \[ \int \ln (x) \, dx \]
8. \[ \int (\ln(x))^2 \, dx \]

*Hint: Use your result from exercise 7.*

9. \[ \int \arctan(x) \, dx \]

10. \[ \int x \arctan(x) \, dx \]

*Hint: Use your result from exercise 9 and you may also find it useful to use the fact that*

\[ \frac{x^2}{1 + x^2} = 1 - \frac{1}{1 + x^2}. \]

11. \[ \int e^{3x} \cos(4x) \, dx \]

### 5 Partial Fraction Decomposition

To evaluate integrals of the form

\[ \int \frac{p(x)}{q(x)} \, dx \]

where \( p \) and \( q \) are polynomials with the degree of \( p \) less than the degree of \( q \), we sometimes first need to write \( p(x)/q(x) \) as a sum of rational functions whose denominators are powers of linear and irreducible quadratic factors of \( q \). This is called doing a “partial fraction decomposition” of the integrand. We illustrate this technique by looking at some examples.

**Example 12** Consider the problem of computing the indefinite integral

\[ \int \frac{x + 1}{x^3 + 2x^2 - 3x} \, dx. \]  

*(4)*

*In factored form the integrand is*

\[ \frac{x + 1}{x(x - 1)(x + 3)}. \]
We attempt to decompose this rational function as
\[
\frac{x + 1}{x(x - 1)(x + 3)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 3} \quad (5)
\]
where \(A\), \(B\), and \(C\) are constants to be determined. To determine the appropriate constants, we multiply both sides of equation (5) by the denominator of the left hand side to obtain
\[
x + 1 = A(x - 1)(x + 3) + Bx(x + 3) + Cx(x - 1). \quad (6)
\]
Substitution of \(x = 0\) into (6) gives
\[
0 + 1 = A(0 - 1)(0 + 3) + B(0)(0 + 3) + C(0)(0 - 1)
\]
from which we conclude that \(A = -1/3\).

Substitution of \(x = 1\) into (6) gives
\[
1 + 1 = A(1 - 1)(1 + 3) + B(1)(1 + 3) + C(1)(1 - 1)
\]
which yields that \(B = 1/2\).

Similarly, substitution of \(x = -3\) into (6) gives \(C = -1/6\).

Substituting the values of \(A\), \(B\), and \(C\) which have been found back into (5), we obtain the partial fraction decomposition
\[
\frac{x + 1}{x(x - 1)(x + 3)} = \frac{-1/3}{x} + \frac{1/2}{x - 1} + \frac{-1/6}{x + 3} \quad (7)
\]
which can also be written as
\[
\frac{x + 1}{x(x - 1)(x + 3)} = -\frac{1}{6} \left( \frac{2}{x} - \frac{3}{x - 1} + \frac{1}{x + 3} \right).
\]
The indefinite integral (4) can now be computed:
\[
\int \frac{x + 1}{x^3 + 2x^2 - 3x} \, dx = -\frac{1}{6} \int \left( \frac{2}{x} - \frac{3}{x - 1} + \frac{1}{x + 3} \right) \, dx
\]
\[
= -\frac{1}{6} \left( 2\ln|x| - 3\ln|x - 1| + \ln|x + 3| \right) + C
\]
\[
= -\frac{1}{6} \left( \ln x^2 - \ln |x - 1|^3 + \ln |x + 3| \right) + C
\]
\[
= -\frac{1}{6} \ln \left( \frac{x^2|x + 3|}{|x - 1|^3} \right) + C
\]
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Example 13 Let us write the rational expression

\[ \frac{1}{x(x+1)(x+2)^3} \]

in partial fraction form.

Like the rational expression which was the subject of Example 12, the rational expression

\[ \frac{1}{x(x+1)(x+2)^3} \]

has a denominator which is a product of linear factors. Since the factor \(x+2\) has multiplicity three, the partial fraction decomposition will have the form

\[ \frac{1}{x(x+1)(x+2)^3} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3}. \]  

(8)

Multiplication of both sides of (8) by the denominator of the left hand side yields

\[ 1 = A(x+1)(x+2)^3 + Bx(x+2)^3 + Cx(x+1)(x+2)^3 + Dx(x+1)(x+2) + Ex(x+1). \]  

(9)

When \(x = 0\) is substituted into (9), we obtain

\[ 1 = A(0+1)(0+2)^3, \]

which implies that \(A = 1/8\).

Substitution of \(x = -1\) and \(x = -2\) yield, respectively, that \(B = -1\) and \(E = 1/2\)

Hence, equation (9) becomes

\[ 1 = \frac{1}{8}(x+1)(x+2)^3 - x(x+2)^3 + Cx(x+1)(x+2)^3 + Dx(x+1)(x+2) + \frac{1}{2}x(x+1). \]  

(10)

When \(x = -3\) is substituted into (10), we obtain

\[ 1 = \frac{1}{8}(-3+1)(-3+2)^3 - (-3)(-3+2)^3 + C(-3)(-3+1)(-3+2)^2 + D(-3)(-3+1)(-3+2) + \frac{1}{2}(-3)(-3+1) \]
which simplifies to
\[ C - D = \frac{1}{8}. \] (11)

Substitution of \( x = 1 \) into (10) yields, after simplification,
\[ 3C + D = \frac{27}{8}. \] (12)

Solving equations (11) and (12) simultaneously yields that \( C = \frac{7}{8} \) and \( D = \frac{3}{4} \).

Substitution of the values of \( A, B, C, D, \) and \( E \) which were found into (8) yields the partial fraction decomposition
\[
\frac{1}{x(x + 1)(x + 2)^3} = \frac{1}{8} \left( \frac{1}{x} - \frac{8}{x + 1} + \frac{7}{x + 2} + \frac{6}{(x + 2)^2} + \frac{4}{(x + 2)^3} \right). \] (13)

Example 14 As our final example, let us write the rational expression
\[ \frac{x^4}{x^3 - 1} \]
in partial fraction form.

Since the degree of the polynomial in the numerator is not less than the degree of the polynomial in the denominator, we must first perform long division before attempting a partial fraction decomposition. After performing the long division, we obtain
\[
\frac{x^4}{x^3 - 1} = x + \frac{x}{x^3 - 1}. \] (14)

The second term on the right hand side of (14) can be written in factored form as
\[
\frac{x}{x^3 - 1} = \frac{x}{(x - 1)(x^2 + x + 1)}. \] (15)

Since the factor \( x^2 + x + 1 \) which appears in equation (15) is an irreducible quadratic factor, we attempt to decompose (15) as a sum of the form
\[
\frac{x}{(x - 1)(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}. \] (16)
After multiplication of both sides by the denominator of the left hand side, equation (16) becomes

\[ x = A(x^2 + x + 1) + (Bx + C)(x - 1). \]  \hspace{1cm} (17)

Substitutions of \( x = 1 \), \( x = 0 \), and \( x = -1 \) into (17) yield, respectively, that \( A = 1/3 \), \( C = 1/3 \), and \( B = -1/3 \).

Going back to (14), we obtain the partial fraction decomposition

\[ \frac{x^4}{x^3 - 1} = x + \frac{1}{3} \left( \frac{1}{x - 1} + \frac{-x + 1}{x^2 + x + 1} \right). \]

Exercise 15

1. Obtain partial fraction decompositions of the following rational expressions. Then evaluate the indefinite integral of each expression (if you can).

   (a) \[ \frac{x - 2}{x^2 + 3x + 2} \]

   (b) \[ \frac{x - 2}{x^2 + 4x + 4} \]

   (c) \[ \frac{2}{x(x^2 - x + 2)} \]

   (d) \[ \frac{2}{(x^2 - 2x + 1)(x^2 + 3x + 2)^2} \]

2. Use the results in Examples 13 and 14 to evaluate the indefinite integrals

   \[ \int \frac{1}{x(x + 1)(x + 2)^3} \, dx \]

   and

   \[ \int \frac{x^4}{x^3 - 1} \, dx. \]

   (You might find the second one to be challenging!)