Here are the proofs of the facts stated in problems 1–5. We will call these Propositions 1–5.

**0.1 Proposition 1**

If \(a\), \(b\), and \(c\) are any integers and \(a + b = a + c\), then \(b = c\).

**0.1.1 Proof of Proposition 1**

Let \(a\), \(b\), and \(c\) be integers and suppose that \(a + b = a + c\).

We know that \(a\) has an additive inverse. It is the integer \(-a\).

Since

\[-a + (a + b) = -a + (a + c),\]

the associativity of addition gives us

\[(−a + a) + b = (−a + a) + c.\]

Since \(-a\) is the additive inverse of \(a\), we know that \(-a + a = 0\). This gives us

\[0 + b = 0 + c.\]

However, since 0 is the additive identity of the integers, we know that \(0 + b = b\)

and \(0 + c = c\).

Therefore \(b = c\).

**0.2 Proposition 2**

If \(a\) is any integer, then \(-1 \cdot a = -a\).

**0.2.1 Proof of Proposition 2**

Let \(a\) be an integer.

Since 1 is the multiplicative identity of the integers, we know that \(1 \cdot a = a\).

Now observe that

\[a + (−1 \cdot a) = 1 ∙ a + (−1 ∙ a)\]
and so by the distributive property,

\[ a + (-1 \cdot a) = (1 + (-1)) \cdot a. \]

Since \(-1\) is the additive inverse of 1, we know that \(1 + (-1) = 0\). Therefore

\[ a + (-1 \cdot a) = 0 \cdot a. \]

However, we have already proved (the Proposition 1 in the textbook) that \(0 \cdot a = 0\). Thus

\[ a + (-1 \cdot a) = 0. \]

The above equation tells us that \(-1 \cdot a\) is the additive inverse of \(a\). Thus \(-1 \cdot a = -a\) (because we already know that \(-a\) is the additive inverse of \(a\)).

**Note:** This proof actually relies on the fact that additive inverses are unique, which is something that is not hard to prove. In fact, suppose that \(a\) is an integer and suppose that there is some integer \(b\) that is an additive inverse of \(a\). Then \(a + b = 0\). Then we have

\[ -a + (a + b) = -a + 0 \]

which gives us

\[ (-a + a) + b = -a + 0 \]

which gives us

\[ 0 + b = -a \]

which gives us \(b = -a\).

This proves that each integer \(a\) has a **unique** additive inverse (which is the number \(-a\)).

### 0.3 Proposition 3

If \(a\) is any integer, then \(-(-a) = a\).

#### 0.3.1 Proof of Proposition 3

Let \(a\) be an integer. We want to prove that the additive inverse of \(-a\) is \(a\).

Since \(-a\) is the additive inverse of \(a\), we know that

\[ a + (-a) = 0. \]
By the commutative property of addition, we can write this as

\[-a + a = 0.\]

The above equation tells us that \(a\) is the additive inverse of \(-a\). We also already know that \(-(-a)\) is the additive inverse of \(-a\). Therefore \(-(-a) = a\).

0.4 Proposition 4

If \(a\) and \(b\) are any integers, then \(-a \cdot b = -(ab)\).

0.4.1 Proof of Proposition 4

Let \(a\) and \(b\) be integers. We want to prove that the additive inverse of \(ab\) is \(-a \cdot b\).

Note that

\[ab + (-a \cdot b) = (a + (-a)) \cdot b\]

(by the distributive property) and since \(a + (-a) = 0\), we have

\[ab + (-a \cdot b) = 0 \cdot b.\]

Since \(0 \cdot b = 0\), we obtain

\[ab + (-a \cdot b) = 0.\]

This tells us that \(-a \cdot b\) is the additive inverse of \(ab\). In other words, \(- (ab) = -a \cdot b\).

0.5 Proposition 5

If \(a\), \(b\), and \(c\) are integers with \(a \neq 0\) and \(ab = ac\), then \(b = c\).

0.5.1 Proof of Proposition 5

We need to use the fact that every real number that is not equal to 0 has a multiplicative inverse. Specifically, if \(x\) is any real number with \(x \neq 0\), then there exists a real number \(x^{-1}\) such that \(x^{-1} \cdot x = 1\).

Now suppose that \(a\), \(b\), and \(c\) are integers with \(a \neq 0\) and suppose that \(ab = ac\).
Since all integers are real numbers, we know that $a$ is a real number. Since $a \neq 0$, then there exists a real number $a^{-1}$ such that $a^{-1} \cdot a = 1$.

Since
\[ a^{-1} (ab) = a^{-1} (ac) \]
and since multiplication in associative in the real numbers, we have
\[ (a^{-1} a) b = (a^{-1} a) c. \]

Thus
\[ 1 \cdot b = 1 \cdot c. \]

Since 1 is the multiplicative identity of the integers, we know that $1 \cdot b = b$ and $1 \cdot c = c$.

Therefore $b = c$.

0.6 From the Textbook (page 14) – problems 7, 8, and 9.

7. (a) $335 = 19 \cdot 17 + 12$
(b) $-335 = -20 \cdot 17 + 5$
(c) $21 = 1 \cdot 13 + 8$
(d) $13 = 1 \cdot 8 + 5$

8. We want to prove that if $a$ divides $b$ and $c$ divides $d$, then $ac$ divides $bd$.

Given that $a$ divides $b$, we know that there exists an integer $s$ such that $sa = b$.

Given that $c$ divides $d$, we know that there exists an integer $t$ such that $tc = d$.

Therefore
\[ (sa) (tc) = bd. \]

By the associative property of multiplication, we have
\[ (st) (ac) = bd. \]

Since $st$ is an integer, we see that $ac$ divides $bd$. 
9. Assuming that $q$ is the quotient and $r$ is the remainder when $a$ is divided by $b$, we have $a = qb + r$ and $0 \leq r < b$.

First we consider the case that $a$ is divisible by $b$. In this case, $a = qb$ (and $r = 0$) and it is clear that $-a = -q \cdot b$ (with remainder $0$).

Now we consider the case that $0 < r < b$. We claim that, in this case, when $-a$ is divided by $b$, we have quotient $-(q + 1)$ and remainder $b - r$. To check that this correct, observe that since $a = qb + r$, we have

$$-a = -qb - r = -qb - b + b - r = -(q + 1)b + b - r.$$ 

Also, since $0 < r < b$, we see that $b - r > 0$ (because $r < b$) and also that $b - r < b$ (because $r > 0$).

An example of this idea is given in exercises 7 a and b above where we see that

$$335 = 19 \cdot 17 + 12$$

and

$$-335 = -20 \cdot 17 + 5.$$