Solutions to Section 1.4 Homework Problems
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1. As stated, this problem is not easy. The prime factorization is
   \[ 12347983 = (281) (43943). \]
   I did this using Maple.

2. Suppose that \( a \) and \( b \) are positive integers and that the complete list
   of primes that divide either \( a \) or \( b \) is \( \{p_1, p_2, \ldots, p_n\} \), and that
   \[ a = p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}, \]
   and
   \[ b = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}. \]
   (a) \( \gcd(a, b) = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n} \)
   where \( r_i = \min (m_i, k_i) \) for \( i = 1 \ldots n \).
   (b) For
   \[ a = 2^5 3^2 5^0 7^1 11^2 13^3 \]
   and
   \[ b = 2^2 3^5 5^1 7^2 11^0 13^0, \]
   we have
   \[ \gcd(a, b) = 2^2 3^2 5^0 7^1 11^0 13^0 = 252. \]

3. Suppose that \( a \) and \( b \) are positive integers and that the complete list
   of primes that divide either \( a \) or \( b \) is \( \{p_1, p_2, \ldots, p_n\} \), and that
   \[ a = p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}, \]
   and
   \[ b = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}. \]
   (a) \( \text{lcm}(a, b) = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n} \)
   where \( r_i = \max (m_i, k_i) \) for \( i = 1 \ldots n \).
   (b) For
   \[ a = 2^5 3^2 5^0 7^1 11^2 13^3 \]
   and
   \[ b = 2^2 3^5 5^1 7^2 11^0 13^0, \]
   we have
   \[ \text{lcm}(a, b) = 2^5 3^5 5^1 7^2 11^2 13^3 = 5.0645 \times 10^{11}. \]
4. Suppose that $p$ is a prime number of the form $p = 3m + 1$ for some natural number $m$.

Since $p$ must be odd (since it is prime and greater than 2), then $p - 1$ must be even. This means that $3m$ must be even (and hence divisible by 2). Therefore, there must exist a natural number $n$ such that $3m = 3(2n) = 6n$. This proves that $p = 6n + 1$ for some natural number $n$.

5. Suppose that $a$ is composite and suppose that every prime that divides $a$ is greater than $p$. Then there are prime numbers $p_1, p_2, \ldots, p_n$ and exponents $m_1, m_2, \ldots, m_n$, all greater than or equal to 1, such that

$$a = p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}$$

where $p_i > \sqrt{a}$ for all $i = 1 \ldots n$. Also, the fact that $a$ is composite implies that $n \geq 2$. This implies that

$$a^2 = (p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n})^2 = p_1^{2m_1} p_2^{2m_2} \cdots p_n^{2m_n} \geq p_1^{2m_1} p_2^{2m_2} > (\sqrt{a})^{2m_1} (\sqrt{a})^{2m_2} = a^{m_1} a^{m_2}.$$ 

Since $a^2 > a^{m_1 + m_2}$ and $m_1 + m_2 \geq 2$, we obtain $a^2 > a^2$, which is impossible. Therefore, it must be the case that at least one of the primes, $p$, that divides $a$ must satisfy $p \leq \sqrt{a}$.

Let us try to determine if the number 541 is prime. Since $\sqrt{541} \approx 23.259$, we know that if 542 is composite, then at least one of the primes, $p$, that divides 541 must satisfy $p \leq 23$. Testing the divisibility of 541 by all such primes we obtain $541 \div 2 = 270.5$, $541 \div 3 = 180.33$, $541 \div 5 = 108.2$, $541 \div 7 = 77.286$, $541 \div 11 = 49.182$, $541 \div 13 = 41.615$, $541 \div 17 = 31.824$, $541 \div 19 = 28.474$, $541 \div 23 = 23.522$. Since 541 is not divisible by any prime that is less than or equal to 23, we conclude that 541 is prime.

6. Using the Sieve of Eratosthenes, one we have reached a step where $p \geq n/2$, the next number to be crossed out is a number $q$ where $q \geq 2 \left(\frac{n}{2}\right) = n$. No more numbers that are less than $n$ will be crossed out after this step. Therefore we have already found all primes between 2 and $n$ at this stage.