1. If we take \( f \) and \( g \) to be the constant polynomials \( f(x) = 0 \) and \( g(x) = 3 \), then these are not equal in \( \mathbb{Z}_5[x] \) (because 0 \( \not\equiv 3 \) in \( \mathbb{Z}_5 \)) but they are equal in \( \mathbb{Z}_3[x] \).

2. The polynomials of degree 2 in \( \mathbb{Z}_2[x] \) are those of the form \( p(x) = ax^2+bx+c \) where \( a, b, \) and \( c \in \mathbb{Z}_2 \) and \( a \neq 0 \). Since 0 and 1 are the only members of \( \mathbb{Z}_2 \), it must be the case that \( a = 1 \). (Thus all polynomials of degree 2 in \( \mathbb{Z}_2[x] \) are monic.) Since there are two possible choices for the value of \( b \) and two possible choices for the value of \( c \), there are a total of 4 polynomials of degree 2 in \( \mathbb{Z}_2[x] \). By similar reasoning, there are a total of \( 4 \cdot 5 \cdot 5 = 100 \) polynomials of degree 2 in \( \mathbb{Z}_5[x] \) (25 of which are monic) and, in general, there are total of \( (n - 1) \cdot n^2 \) polynomials of degree 2 in \( \mathbb{Z}_n[x] \), of which \( n^2 \) of them are monic.

3. Let

\[
\begin{align*}
p(x) &= 2x^2 + 3x + 1 \\
q(x) &= 3x + 2
\end{align*}
\]

in \( \mathbb{Z}_5[x] \). Then

\[
p(x) + q(x) = 1
\]

and

\[
p(x) \cdot q(x) = (2x^2 + 3x + 1)(3x + 2) = x^3 + 4x^2 + 4x^2 + x + 3x + 2 = x^3 + 3x^2 + 4x + 2.
\]

4. This follows from the fact that if \( a \equiv b \mod m \) and \( c \equiv d \mod m \), then \( (a + c) \equiv (b + d) \mod m \) and \( ac \equiv bd \mod m \). We will omit writing the details of this. (In other words, skip this homework problem.)

5. (a)

\[
\begin{align*}
x^3 - 1 &= x^2(x - 1) + x^2 - 1 \\
x^2 - 1 &= x(x - 1) + x - 1 \\
x - 1 &= 1(x - 1) + 0.
\end{align*}
\]
Thus
\[ x^3 - 1 = x^2 (x - 1) + x (x - 1) + 1 (x - 1) + 0 = (x^2 + x + 1) (x - 1) + 0 \]
and we see that \( q(x) = x^2 + x + 1 \) and \( r(x) = 0 \).

(b) In general
\[ x^n - 1 = (x^{n-1} + x^{n-2} + \cdots + 1) (x - 1) + 0. \]

(c) In \( Z_5 \)[x],
\[
\begin{align*}
2x^4 + x^3 + 3x + 1 &= 2x^2 (x^2 + x + 1) + 4x^3 + 3x^2 + 3x + 1 \\
4x^3 + 3x^2 + 3x + 1 &= 4x (x^2 + x + 1) + 4x^2 + 4x + 1 \\
4x^2 + 4x + 1 &= 4 (x^2 + x + 1) + 2.
\end{align*}
\]
Thus
\[
2x^4 + x^3 + 3x + 1 = (2x^2 + 4x + 4) (x^2 + x + 1) + 2
\]
and we see that \( q(x) = 2x^2 + 4x + 4 \) and \( r(x) = 2 \). Let us check to see if we did this correctly:
\[
(2x^2 + 4x + 4) (x^2 + x + 1) + 2 = 2x^4 + 2x^3 + 2x^2 + 4x^3 + 4x^2 + 4x + 2x^2 + 4x + 4 + 2
\]
\[
= 2x^4 + x^3 + 3x + 1.
\]

6. First suppose that \( a \) is a zero divisor in \( R \). Then \( a \neq 0 \) and there exists \( b \in R \) with \( b \neq 0 \) such that either \( ab = 0 \) or \( ba = 0 \). In this case the constant polynomial \( f(x) = a \) is a zero divisor in \( R[x] \) because, for the constant polynomial \( g(x) = b \), we have either \( f(x) g(x) = O(x) \) or \( g(x) f(x) = O(x) \). (Recall that we are using \( O(x) \) to denote the constant polynomial \( O(x) = 0 \).)

Now suppose that \( R \) does not have any zero divisors and let \( f(x) \) be a non–zero polynomial in \( R[x] \). Since \( f \neq O \), we know that \( f \) has the form \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \) where \( n \geq 0 \) and \( a_n \neq 0 \). If we take any other non–zero polynomial in \( R[x] \), say \( g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0 \) where \( m \geq 0 \) and \( b_m \neq 0 \), then the leading term in \( f(x) g(x) \) is \( a_n b_m x^{n+m} \) and since \( a_n b_m \neq 0 \) (because \( R \) has no zero divisors), we see that \( fg \neq O \). Thus \( R[x] \) has no zero divisors.

We have proved that \( R[x] \) has no zero divisors if and only if \( R \) has no zero divisors.
7. \[
\begin{align*}
x^5 - x^3 + x^2 - 2x + 1 &= x^3(x^2 + 1) - 2x^3 + x^2 - 2x + 1 \\
-2x^3 + x^2 - 2x + 1 &= -2x(x^2 + 1) + x^2 + 1 \\
x^2 + 1 &= (x^2 + 1) + 0
\end{align*}
\]
give us
\[
x^5 - x^3 + x^2 - 2x + 1 = (x^3 - 2x + 1)(x^2 + 1)
\]
and we see that \(x^5 - x^3 + x^2 - 2x + 1\) is divisible by \(x^2 + 1\) in \(Q[x]\).

8. Suppose we have polynomials \(g(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0\) and \(p(x) = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_0\) (where \(n \geq m\)) in \(R[x]\) where \(R\) is some ring and suppose that \(b_m\) is a unit in \(R\). Then \(b_m^{-1}\) exists in \(R\) and we have
\[
a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 = a_nb_m^{-1}x^{n-m} (b_mx^m + b_{m-1}x^{m-1} + \cdots + b_0) + r_1(x).
\]
Since the next step in the Division Algorithm process involves division of \(r_1(x)\) by \(p(x)\) and since \(b_m\) is a unit, we see that we can continue the process as many times as needed until we arrive at a final remainder.

9. For \(g(x) = x^{15} + x^7 - 3x^2 + 1\) and \(p(x) = x - 1\), we have \(g(1) = 0\). This means that \(g(x)\) is divisible by \(p(x)\) and hence that the remainder is
zero. We can see this directly also:

\[
x^{15} + x^7 - 3x^2 + 1 = x^{14} (x - 1) + x^{14} + x^7 - 3x^2 + 1
\]
\[
x^{14} + x^7 - 3x^2 + 1 = x^{13} (x - 1) + x^{13} + x^7 - 3x^2 + 1
\]
\[
x^{13} + x^7 - 3x^2 + 1 = x^{12} (x - 1) + x^{12} + x^7 - 3x^2 + 1
\]
\[
x^{12} + x^7 - 3x^2 + 1 = x^{11} (x - 1) + x^{11} + x^7 - 3x^2 + 1
\]
\[
x^{11} + x^7 - 3x^2 + 1 = x^{10} (x - 1) + x^{10} + x^7 - 3x^2 + 1
\]
\[
x^{10} + x^7 - 3x^2 + 1 = x^9 (x - 1) + x^9 + x^7 - 3x^2 + 1
\]
\[
x^9 + x^7 - 3x^2 + 1 = x^8 (x - 1) + x^8 + x^7 - 3x^2 + 1
\]
\[
x^8 + x^7 - 3x^2 + 1 = x^7 (x - 1) + 2x^7 - 3x^2 + 1
\]
\[
2x^7 - 3x^2 + 1 = 2x^6 (x - 1) + 2x^6 - 3x^2 + 1
\]
\[
2x^6 - 3x^2 + 1 = 2x^5 (x - 1) + 2x^5 - 3x^2 + 1
\]
\[
2x^5 - 3x^2 + 1 = 2x^4 (x - 1) + 2x^4 - 3x^2 + 1
\]
\[
2x^4 - 3x^2 + 1 = 2x^3 (x - 1) + 2x^3 - 3x^2 + 1
\]
\[
2x^3 - 3x^2 + 1 = 2x^2 (x - 1) - x^2 + 1
\]
\[
-x^2 + 1 = -x (x - 1) - x + 1
\]
\[
x + 1 = -1 (x - 1) + 0
\]
\[
gives
\]
\[
x^{15} + x^7 - 3x^2 + 1 = (x^{14} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + 2x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2
\]

10. In \( \mathbb{Z}_5 [x] \), if we define \( g (x) = x^4 + 3x + 2 \) and \( p (x) = x + 3 \), we see (since \(-3 = 2\)) that \( p (x) = x - 2 \). Since \( g (2) = 2^4 + 3 \cdot 2 + 2 = 1 + 1 + 2 = 4 \), the remainder upon division of \( g (x) \) by \( p (x) \) must be 4. Let us check this directly:

\[
x^4 + 3x + 2 = x^3 (x + 3) - 3x^3 + 3x + 2
\]
\[
-3x^3 + 3x + 2 = -3x^2 (x + 3) + 9x^2 + 3x + 2
\]
\[
9x^2 + 3x + 2 = 9x (x + 3) - 24x + 2
\]
\[
-24x + 2 = -24 (x + 3) + 74.
\]

This gives us

\[
x^4 + 3x + 2 = (x^3 - 3x^2 + 9x - 24) (x + 3) + 74,
\]
but we are working in $\mathbb{Z}_5[x]$, so this is the same as
\[
x^4 + 3x + 2 = (x^3 + 2x^2 + 4x + 1)(x + 3) + 4.
\]

11. In $\mathbb{Z}_6[x]$, $f(x) = x^2 + x$ has 0, 2, and 3 as roots.

12. Of course, $f(x) = 0$ has roots (0 and 1) in $\mathbb{Z}_2$. $f(x) = 1$ does not have roots in $\mathbb{Z}_2$. Let us look at the degree one polynomials in $\mathbb{Z}_2[x]$: They are $f_1(x) = x$ and $f_2(x) = x + 1$. Since $f_1(0) = 0$ and $f_2(1) = 0$, both of these have roots in $\mathbb{Z}_2$. Now let us look at the degree 2 polynomials in $\mathbb{Z}_2[x]$: They are $f_1(x) = x^2$, $f_2(x) = x^2 + 1$, $f_3(x) = x^2 + x$, and $f_4(x) = x^2 + x + 1$. All of these have roots in $\mathbb{Z}_2$ except for $f_4(x)$. After looking at these few examples, I think that we can see that the answer to this question is that any polynomial in $\mathbb{Z}_2[x]$ whose constant term is 0 has a root (0) in $\mathbb{Z}_2$ and any polynomial in $\mathbb{Z}_2[x]$ whose constant term is 1 and whose degree is odd has a root (1) in $\mathbb{Z}_2$. The other polynomials in $\mathbb{Z}_2$ do not have roots in $\mathbb{Z}_2$.

13. This can be done by examining each of the quadratic polynomials in $\mathbb{Z}_3[x]$. (There are not too many of them.)