1. $3x^3 + 2x + 5$ is primitive and $6x^4 + 3x + 9$ is not primitive.

2. Let $p(x) \in \mathbb{Q}[x]$. (We may assume that the coefficients of $p(x)$ are written in lowest terms.) Let $m$ be the least common multiple of the denominators of the coefficients of $p(x)$. Then $mp(x) \in \mathbb{Z}[x]$. Now let $c$ be the content of $mp(x)$. Then $mp(x) = cq(x)$ where $q(x) \in \mathbb{Z}[x]$ is primitive. Also $p(x) = (c/m)q(x)$. Since $m$ and $c$ are integers, then $c/m$ is a rational number. We have shown that there exists a rational number, $t$, and a primitive polynomial $q(x) \in \mathbb{Z}[x]$ such that $p(x) = tq(x)$.

3. Suppose that $f(x) \in \mathbb{Z}[x]$, $g(x) \in \mathbb{Z}[x]$ and suppose that $f(x)$ is not primitive. Let $c$ be the content of $f(x)$. Then $f(x) = ch(x)$ where $c$ is an integer, $c \neq 1$, and $h(x)$ is primitive. Let $d$ be the content of $g(x)$. Then $g(x) = dk(x)$ where $d$ is an integer and $k(x)$ is primitive. Now we have

$$f(x)g(x) = cdh(x)k(x).$$

Since $h(x)$ and $k(x)$ are both primitive, then $h(x)k(x)$ is also primitive by Gauss’s Lemma. Thus the content of $f(x)g(x)$ is $cd$ and, since $c \neq 1$, we know that $cd \neq 1$. Therefore $f(x)g(x)$ is not primitive.

We have proved that if $f(x)$ is not primitive, then $f(x)g(x)$ is not primitive. By entirely similar reasoning, we could prove that if $g(x)$ is not primitive, then $f(x)g(x)$ is not primitive. By contrapositive, if $f(x)g(x)$ is primitive, then both $f(x)$ and $g(x)$ must be primitive. Note that what we have proved is the converse of Gauss’s Lemma. (This converse is much easier to prove than Gauss’s Lemma itself which was why it was left as a homework exercise.)

4. (a) Since $f(x) = x^4 + x + 1 \in \mathbb{Z}[x]$, then we know, by Theorem 2 on page 131, that if $f(x)$ is reducible in $\mathbb{Q}[x]$, it must also be reducible in $\mathbb{Z}[x]$. Since $f(x)$ has degree 4, if $f(x)$ is non-trivially factorable in $\mathbb{Z}[x]$, then it must be the product of a linear and a cubic polynomial or the product of two quadratic polynomials in
Let us consider both possibilities. The first possibility is that
\[ f(x) = (ax + b)(cx^3 + dx^2 + ex + f) \]
where the coefficients are all integers. This gives
\[ x^4 + x + 1 = acx^4 + adx^3 + aex^2 + afx + bcx^3 + bdx^2 + bex + bf \]
\[ = acx^4 + (ad + bc)x^3 + (ae + bd)x^2 + (af + be)x + bf \]
which gives
\[
\begin{align*}
ac &= 1 \\
ad + bc &= 0 \\
ae + bd &= 0 \\
af + be &= 1 \\
bf &= 1.
\end{align*}
\]
Since \(ac = 1\) and \(a\) and \(c\) are integers, then either \(a = c = 1\) or \(a = c = -1\). Likewise, either \(b = f = 1\) or \(b = f = -1\). Let us consider the possibility that \(a = c = 1\) and \(b = f = 1\). This gives us
\[
\begin{align*}
d + 1 &= 0 \\
e + d &= 0 \\
1 + e &= 1
\end{align*}
\]
which gives \(e = 0, d = 0, d = -1\) (an obvious contradiction). Next we consider the possibility \(a = c = 1, b = f = -1\). This gives
\[
\begin{align*}
d - 1 &= 0 \\
e - d &= 0 \\
-1 - e &= 1
\end{align*}
\]
which implies \(e = -2, d = -2, d = 1\) (a contradiction). The possibility \(a = c = -1, b = f = 1\) gives
\[
\begin{align*}
d - 1 &= 0 \\
e + d &= 0 \\
-1 + e &= 1
\end{align*}
\]
which gives \( e = 2, \ d = 2, \ d = -1 \) (a contradiction) and the possibility \( a = c = -1, \ b = f = -1 \) gives
\[
\begin{align*}
-d + 1 &= 0 \\
-e - d &= 0 \\
1 - e &= 1
\end{align*}
\]
which also gives the contradictory \( e = 0, \ d = 0, \ d = 1 \). We therefore see that \( f(x) \) cannot be the product of a linear and a cubic polynomial in \( \mathbb{Z}[x] \).

Let us now check out the possibility that
\[
x^4 + x + 1 = (ax^2 + bx + c) \left(dx^2 + ex + f\right)
= adx^4 + (ae + bd)x^3 + (af + cd + be)x^2 + (bf + ce)x + cf.
\]
In this case,
\[
\begin{align*}
ad &= 1 \\
ae + bd &= 0 \\
af + cd + be &= 0 \\
bf + ce &= 1 \\
cf &= 1.
\end{align*}
\]
Consider the case \( a = d = 1, \ c = f = 1 \) which gives
\[
\begin{align*}
e + b &= 0 \\
1 + 1 + be &= 0 \\
b + e &= 1.
\end{align*}
\]
The contradiction here is \( e + b = 0, \ e + b = 1 \).
The case \( a = d = 1, \ c = f = -1 \) gives
\[
\begin{align*}
e + b &= 0 \\
-2 + be &= 0 \\
-b - e &= 1
\end{align*}
\]
which is contradictory.
The case \( a = d = -1, c = f = 1 \) gives

\[
-e - b = 0 \\
-2 + be = 0 \\
b + e = 1
\]

which is contradictory.

The case \( a = d = -1, c = f = -1 \) gives

\[
-e - b = 0 \\
2 + be = 0 \\
-b - e = 1
\]

which is contradictory.

We conclude that \( f (x) \) is irreducible in \( \mathbb{Z} [x] \) and in \( \mathbb{Q} [x] \).

(b) Since

\[
x^3 - \frac{1}{2} x^2 - \frac{1}{2} = \frac{1}{2} (2x^3 - x^2 - 1)
\]

we see that \( x^3 - \frac{1}{2} x^2 - \frac{1}{2} \) is reducible in \( \mathbb{Q} [x] \) if and only if \( 2x^3 - x^2 - 1 \) is reducible in \( \mathbb{Z} [x] \). It is easily observed that 1 is a root of \( 2x^3 - x^2 - 1 \). Thus \( 2x^3 - x^2 - 1 \) is reducible in \( \mathbb{Z} [x] \). Therefore \( x^3 - \frac{1}{2} x^2 - \frac{1}{2} \) is reducible in \( \mathbb{Q} [x] \).

5. (a) For \( f (x) = x^3 + x + 1 \) in \( \mathbb{Z} [x] \), we have \( f_2 (x) = x^3 + x + 1 \) in \( \mathbb{Z}_2 [x] \). Note that \( \deg (f_2 (x)) = \deg (f (x)) \). Since neither 0 nor 1 is a root of \( f_2 (x) \) and \( f_2 (x) \) has degree 2, we conclude that \( f_2 (x) \) is irreducible in \( \mathbb{Z}_2 [x] \) and hence that \( f (x) \) is irreducible in \( \mathbb{Z} [x] \).

(b) For \( f (x) = x^4 + x^2 + x + 1 \) in \( \mathbb{Z} [x] \), we have \( f_3 (x) = x^4 + x^2 + x + 1 \) in \( \mathbb{Z}_3 [x] \). Note that \( \deg (f_3 (x)) = \deg (f (x)) \). Also

\[
f_3 (0) = 1 \\
f_3 (1) = 1 \\
f_3 (2) = 1 + 1 + 1 + 1 = 1
\]

so \( f_3 (x) \) has no linear factors in \( \mathbb{Z}_3 [x] \). Possible quadratic factors
of $f_3 (x)$ in $Z_3 [x]$ are

$$
x^2 \\
x^2 + 1 \\
x^2 + x \\
x^2 + x + 1 \\
x^2 + 2 \\
x^2 + 2x \\
x^2 + 2x + 2 \\
x^2 + x + 2 \\
x^2 + 2x + 1
$$

and

$$
2x^2 \\
2x^2 + 1 \\
2x^2 + x \\
2x^2 + x + 1 \\
2x^2 + 2 \\
2x^2 + 2x \\
2x^2 + 2x + 2 \\
2x^2 + x + 2 \\
2x^2 + 2x + 1.
$$

However, the only ones of these eighteen possibilities that do not have roots in $Z_3$ are

$$
x^2 + 1 \\
x^2 + 2x + 2 \\
x^2 + x + 2 \\
2x^2 + x + 1 \\
2x^2 + 2 \\
2x^2 + 2x + 1.
$$
Also
\[ 2x^2 + 2 = 2(x^2 + 1) \]
\[ 2x^2 + x + 1 = 2(x^2 + 2x + 2) \]
\[ 2x^2 + 2x + 1 = 2(x^2 + x + 2) \]

so the only three possibilities that we need to check are \( x^2 + 1 \), \( x^2 + 2x + 2 \), and \( x^2 + x + 2 \). For each, we use the Division Algorithm.

\[ x^4 + x^2 + x + 1 = x^2(x^2 + 1) + x + 1 \]

shows that \( f_3(x) \) is not divisible by \( x^2 + 1 \).

\[ x^4 + x^2 + x + 1 = x^2(x^2 + 2x + 2) + x^3 + 2x^2 + x + 1 \]
\[ x^3 + 2x^2 + x + 1 = x(x^2 + 2x + 2) + 2x + 1 \]

shows that \( f_3(x) \) is not divisible by \( x^2 + 2x + 2 \).

\[ x^4 + x^2 + x + 1 = x^2(x^2 + x + 2) + 2x^3 + 2x^2 + x + 1 \]
\[ 2x^3 + 2x^2 + x + 1 = 2x(x^2 + x + 2) + 1 \]

shows that \( f_3(x) \) is not divisible by \( x^2 + x + 2 \). Therefore \( f_3(x) \) is irreducible in \( \mathbb{Z}_3[x] \) and this tells us that \( f(x) \) is irreducible in \( \mathbb{Z}[x] \).

6. First we recall a homework exercise from earlier in the course (which was also on the first exam) that states that if \( a, b, \) and \( c \) are integers with \( \gcd(a, b) = 1 \) and \( \gcd(a, c) = 1 \), then \( \gcd(a, bc) = 1 \). In particular, it follows from this that if \( \gcd(a, b) = 1 \), then \( \gcd(a, b^n) = 1 \) for any integer \( n \geq 0 \).

Now, suppose that

\[ f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \]

is in \( \mathbb{Z}[x] \) and suppose that \( s/t \) with \( \gcd(s, t) = 1 \) is a rational root of \( f(x) \). Then

\[ a_n\left(\frac{s}{t}\right)^n + a_{n-1}\left(\frac{s}{t}\right)^{n-1} + \cdots + a_1\left(\frac{s}{t}\right) + a_0 = 0 \]
which gives us
\[ a_n s^n + a_{n-1} s^{n-1} t + \cdots + a_1 s t^{n-1} + a_0 t^n = 0. \]

Since
\[ a_0 t^n = - (a_n s^n + a_{n-1} s^{n-1} t + \cdots + a_1 s t^{n-1}) \]
and since the right hand side is divisible by \( s \), then the left hand side must also be divisible by \( s \). However, \( \gcd(s, t^n) = 1 \), so Euclid’s Lemma tells us that \( a_0 \) must be divisible by \( s \). By similar reasoning, Euclid’s Lemma tells us that \( a_n \) must be divisible by \( t \). This proves Descarte’s Criterion.

(a) If \( s/t \) is a rational root of \( x^3 - x + 1 \) (in lowest terms), then the possibilities for \( s \) are \( \pm 1 \) and the possibilities for \( t \) are \( \pm 1 \). Therefore the possible rational roots are \( \pm 1 \). Neither of these is a root so this polynomial has no rational roots.

(b) For \( 2x^3 + x - 1 \), possibilities are \( s = \pm 1, t = \pm 1, \pm 2, s/t = \pm 1, \pm 1/2 \). Checking:
\[
\begin{align*}
f(1) &= 2 + 1 - 1 = 2 \\
f(-1) &= -2 - 1 - 1 = -4 \\
f\left(\frac{1}{2}\right) &= 2 \left(\frac{1}{2}\right)^3 + \frac{1}{2} = -1 \\
f\left(-\frac{1}{2}\right) &= 2 \left(-\frac{1}{2}\right)^3 - \frac{1}{2} = -\frac{7}{4} 
\end{align*}
\]
so this polynomial has no rational roots.

(c) For \( 2x^3 - x^2 + 2x - 1 \), possibilities are \( s = \pm 1, t = \pm 1, \pm 2, s/t = \pm 1, \pm 1/2 \). Checking:
\[
\begin{align*}
f(1) &= 2 - 1 + 2 - 1 = 2 \\
f(-1) &= -2 - 1 - 2 - 1 = -6 \\
f\left(\frac{1}{2}\right) &= 2 \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^2 + 2 \left(\frac{1}{2}\right) - 1 = 0 \\
f\left(-\frac{1}{2}\right) &= 2 \left(-\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^2 + 2 \left(-\frac{1}{2}\right) - 1 = -\frac{5}{2} 
\end{align*}
\]
so the only rational root of \( f(x) \) is \( x = 1/2 \).
(d) For $6x^4 + x^3 + 4x^2 + x - 2$, possibilities are $s = \pm 1, \pm 2, t = \pm 1, \pm 2, \pm 3, \pm 6, s/t = \pm 1, \pm 1/2, \pm 1/3, \pm 1/6 \pm 2, \pm 2/3$. We now check: $f(1) = 10, f(-1) = 6, f(1/2) = 0, f(-1/2) = -5/4, f(1/3) = -10/9, f(-1/3) = -50/27, f(1/6) = -185/108, f(-1/6) = -37/18, f(2) = 120, f(-2) = 100, f(2/3) = 52/27, and f(-2/3) = 0. Therefore the only rational roots of $f(x)$ are $x = 1/2$ and $x = -2/3$.

7. (a) $f(x) = 2x^7 + 5x^3 - 25x + 15$ is irreducible by Eisenstein’s Criterion with $p = 5$.
(b) $g(x) = x^4 + 3x^2 + 2 = (x^2 + 2)(x^2 + 1)$.
(c) $h(x) = x^5 + 2x^3 + 2x^2 + 2$ is irreducible by Eisenstein’s Criterion with $p = 2$.
(d) $k(x) = x^4 + 1$ has no roots in $\mathbb{Q}[x]$. If $k(x)$ is reducible in $\mathbb{Q}[x]$, then it must be reducible in $\mathbb{Z}[x]$, so it must be the product of two quadratic polynomials.

$$x^4 + 1 = (ax^2 + bx + c)(dx^2 + ex + f)$$

$$= adx^4 + (ae + bd)x^3 + (af + cd + be)x^2 + (bf + ce)x + cf.$$ 

gives

$$ad = 1$$
$$ae + bd = 0$$
$$af + cd + be = 0$$
$$bf + ce = 0$$
$$cf = 1.$$ 

We have four possibilities to check.
Case 1: $a = d = 1$ and $c = f = 1$ gives

$$e + b = 0$$
$$2 + be = 0$$
$$b + e = 0$$

which gives $b = -e$ and thus $e^2 = 2$ and thus $e = \sqrt{2}$, but $\sqrt{2}$ is not a rational number.
Case 2: \( a = d = 1 \) and \( c = f = -1 \) gives

\[
\begin{align*}
e + b &= 0 \\
-2 + be &= 0 \\
-b - e &= 0
\end{align*}
\]

which gives \( b = -e \) and thus \( e^2 = -2 \) which is not possible for \( e \in Q \).

Case 3: \( a = d = -1 \) and \( c = f = 1 \) gives

\[
\begin{align*}
-e - b &= 0 \\
-2 + be &= 0 \\
b + e &= 0
\end{align*}
\]

which gives a contradiction.

Case 4: \( a = d = -1 \) and \( c = f = -1 \) gives

\[
\begin{align*}
-e - b &= 0 \\
2 + be &= 0 \\
-b - e &= 0
\end{align*}
\]

which gives a contradiction.

We conclude that \( k(x) = x^4 + 1 \) is irreducible in \( Q[x] \).