1. Factor the polynomial
   \[ a(x) = 3x^3 + 4x^2 + 3 \]
as completely as possible in \( \mathbb{Z}_5 \). Show all work.

2. Explain why each of the following polynomials is irreducible in \( \mathbb{Q}[x] \).
   a. \( a(x) = x^2 - 2 \)
   b. \( a(x) = 3x^3 + 4x^2 + 3 \)
   c. \( a(x) = 3x^5 + 7x^4 - 70x^3 - 21x^2 + 63x - 63 \)
   d. \( a(x) = x^4 + 1 \)

3. Suppose that \( E \) is a field and \( F \) is a subfield of \( E \) and \( c \in E \). Define
   \[ J_c = \{ a(x) \in F[x] \mid a(c) = 0 \} \]
   Prove that \( J \) is an ideal in \( F[x] \).

4. Suppose that \( E \) is a field and that \( F \) is a subfield of \( E \) and \( c \in E \). Define what it means for \( c \) to be algebraic over \( F \).

5. Show that each of the following numbers is algebraic over \( \mathbb{Q} \). (Here we are taking \( E = \mathbb{C} \) and \( F = \mathbb{Q} \).)
   a. \( \frac{2}{3} + i \)
   b. \( \sqrt{2} + \sqrt{3} \)
   c. \( \sqrt{1 + \sqrt{1 + \sqrt{2}}} \).

6. Suppose that \( E \) is a field and that \( F \) is a subfield of \( E \) and \( c \in E \) with \( c \neq 0 \). Prove that if \( c \) is algebraic over \( F \) then \( c^{-1} \) is also algebraic over \( F \).

7. Find a basis for \( \mathbb{Q}(\sqrt{5}, i) \) over \( \mathbb{Q} \) and describe the elements of \( \mathbb{Q}(\sqrt{5}, i) \).
   (Note: The first step is to find the minimum polynomial of \( \sqrt{5} i \) over \( \mathbb{Q} \). It is a bit of a painstaking exercise to prove that this polynomial is irreducible so you don’t have to do this. Just find the polynomial and use the fact that it is of degree 6 to continue the problem.)

8. Find a basis for \( \mathbb{Q}(\sqrt{5}, i) \) over \( \mathbb{Q} \) and describe the elements of \( \mathbb{Q}(\sqrt{5}, i) \).
   Comparing your results to the results of question 6, what do you observe?

9. a. Explain why the polynomial \( p(x) = x^3 + x^2 + 2x + 1 \) is irreducible in \( \mathbb{Z}_3[x] \).
   b. Explain how you can use the fact from part a to construct a field that contains exactly 27 elements. (Do not actually construct the field. Just explain in some detail how it would be done using the Basic Field Extension Theorem.)