The Bolzano – Weierstrass Theorem

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The Bolzano–Weierstrass Theorem (BW Theorem), named after Bernard Bolzano (1781–1848) and Karl Weierstrass (1815–1897), tells us that if a subset, $A$, of $R$ is bounded and has infinitely many members, then $A$ must have at least one limit point. An equivalent way of stating the theorem (in terms of sequences rather than sets) is that a bounded sequence must have a convergent subsequence. We have already used the BW Theorem in this course in proving theorems and working problems that required us to obtain a convergent subsequence from a bounded sequence. The proof (of the sequential version of the theorem) was a homework exercise (problem 18 in Section 1.3 of the textbook) but since this theorem is of great importance we should devote some time to studying it more closely. First, we recall some definitions of relevant concepts.

**Definition 1** A set, $A$, is said to be infinite if $A$ has infinitely many members.

**Definition 2** A subset, $A$, of $R$ is said to be bounded if there exists $M > 0$ such that $|x| \leq M$ for all $x \in A$.

**Remark 3** Previously, we defined $A$ to be bounded if $A$ is bounded above and below. Definition 2 is equivalent.

**Definition 4** If $A \subseteq R$, then a real number, $a$, is said to be a limit point of $A$ if for every $\delta > 0$, the set $(a - \delta, a + \delta) - \{a\}$ contains at least one member of $A$.

**Example 5** The set $A = [2, 6]$ is infinite, bounded, and every member of $A$ is a limit point of $A$. Also, $A$ has no other limit points.
Example 6 The set $A = \{1, 3, 7, 13\}$ is finite (having only four members) and bounded. $A$ has no limit points.

Example 7 The set $A = \{1/2, 2/3, 3/4, 4/5, \ldots \}$ is infinite and bounded. The only limit point of $A$ is 1.

Exercise 8 For each of the following sets, $A$, decide whether $A$ is finite or infinite, bounded or unbounded, and find all limit points of $A$.

1. $A = [-4, \infty)$
2. $A = (-4, 2)$
3. $A = (0, 1) \cup (1, 2)$
4. $A = (0, 1) \cup \{2\}$
5. $A = \{x \in [0, 1] \mid x \text{ is rational}\}$
6. $A = \mathbb{N}$ (the set of all natural numbers)
7. $A = \{-2, 12, 47\}$

The following lemma gives another (seemingly stronger but really equivalent) characterization of limit points.

Lemma 9 If $A \subseteq \mathbb{R}$ and $a$ is a real number, then $a$ is a limit point of $A$ if and only if for every $\delta > 0$ the set $(a - \delta, a + \delta)$ contains infinitely many members of $A$.

Proof. If for every $\delta > 0$ the set $(a - \delta, a + \delta)$ contains infinitely many members of $A$, then for every $\delta > 0$ the set $(a - \delta, a + \delta) - \{a\}$ must contain at least one (and in fact infinitely many) members of $A$ and it is thus clear that $a$ is a limit point of $A$. To prove the converse, we suppose that $a$ is a limit point of $A$. By definition, this means that for every $\delta > 0$ the set $(a - \delta, a + \delta) - \{a\}$ contains at least one member of $A$.

For the sake of obtaining a contradiction, suppose that there exists $\delta_1 > 0$ such the set $(a - \delta_1, a + \delta_1)$ contains only a finite number of members of $A$. Then the set $(a - \delta_1, a + \delta_1) - \{a\}$ must also contain only a finite number of members, $x_1, x_2, \ldots, x_n$, of $A$. Since the set $\{x_1, x_2, \ldots, x_n\}$ is finite and $a$ is not a member of this set, there exists $j \in \{1, 2, \ldots, n\}$ such that $0 <$
\[ |x_j - a| \leq |x_i - a| \text{ for all } i \in \{1, 2, \ldots, n\} \] (that is, the point \( x_j \) is closer to \( a \) than any other \( x_i \)). Hence, if we let \( \delta_2 = |x_j - a|/2 \), then \( \delta_2 > 0 \) and the set \((a - \delta_2, a + \delta_2) - \{a\}\) contains no members of \( A \). This is a contradiction to our supposition that \( a \) is a limit point of \( A \). Hence, for every \( \delta > 0 \), the set \((a - \delta, a + \delta)\) contains infinitely many members of \( A \). ■

**Theorem 10 (BW Theorem – Set Version)** If \( A \) is an infinite and bounded subset of \( R \), then \( A \) has at least one limit point.

To prove Theorem 10, we will actually prove an equivalent theorem (Theorem 17) that states the result in terms of sequences rather than in terms of sets. (It is actually this latter version of the theorem that we have used in some of our past work.) First, we give some relevant definitions.

**Definition 11** A sequence, \( f : N \rightarrow R \), is said to be bounded if the set \( f(N) \) (the range of \( f \)) is bounded. In other words, if \( f \) is a sequence with terms \( a_1, a_2, a_3, \ldots \), then \( f \) is bounded if there exists \( M > 0 \) such that \( |a_n| \leq M \) for all \( n = 1, 2, 3, \ldots \).

**Definition 12** A sequence \( a_1, a_2, a_3, \ldots \) is said to have limit \( L \) (or is said to converge to \( L \)) if for every \( \varepsilon > 0 \) there exists an integer \( m \) such that \( |a_n - L| < \varepsilon \) for all \( n \geq m \).

**Definition 13** A real number, \( L \), is called a cluster point of a sequence \( a_n \) if \( a_n \) has a subsequence that converges to \( L \).

**Example 14** The sequence \( a_n = 1 - 1/n \) is bounded and has limit 1. Hence, every subsequence of \( a_n \) also has limit 1 and 1 is the only cluster point of \( a_n \).

**Example 15** The sequence \( a_n = \cos(n\pi/2) + 1/n \) has terms

\[
\begin{align*}
a_1 &= 0 + 1 = 1 \\
a_2 &= -1 + \frac{1}{2} = -\frac{1}{2} \\
a_3 &= 0 + \frac{1}{3} = \frac{1}{3} \\
a_4 &= 1 + \frac{1}{4} = \frac{5}{4} \\
a_5 &= 0 + \frac{1}{5} = \frac{1}{5} \\
\vdots
\end{align*}
\]
This sequence is bounded but not convergent. However, the subsequence \( b_k = a_{4k} \) has terms

\[
\begin{align*}
  b_1 &= a_4 = 1 + \frac{1}{4} \\
  b_2 &= a_8 = 1 + \frac{1}{8} \\
  b_3 &= a_{12} = 1 + \frac{1}{12} \\
  \vdots
\end{align*}
\]

and this subsequence has limit 1. In addition, \( a_n \) also has subsequences with limit 0 and subsequences with limit \(-1\). All convergent subsequences of \( a_n \) converge either to 1 or to 0 or to \(-1\). Thus, 1, 0, and \(-1\) are the cluster points of \( a_n \).

**Exercise 16** Find all cluster points of each of the following sequences, \( a_n \).

1. \( a_n = 1 + (-1)^n /n \)
2. \( a_n = (-1)^n \)
3. \( a_n = \sin (n\pi/4) \)
4. \( a_n = (-1)^n \sin (n\pi/4) \)
5. \( a_n = \frac{1}{n} \sin (n\pi/4) \)

We now state the sequential version of the Bolzano–Weierstrass Theorem.

**Theorem 17 (BW Theorem – Sequential Version)** If \( a_n \) is a bounded sequence, then \( a_n \) has at least one cluster point.

To conclude, we will show that Theorems 10 and 17 are equivalent and we will prove Theorem 17 (thus really proving both Theorems 17 and 10). To show that Theorems 10 and 17 are equivalent, we must show that the truth of Theorem 10 implies the truth of Theorem 17 and vice-versa.

**Proof that Theorems 10 and 17 are Equivalent:**

Assume that Theorem 10 is true and let \( a_n \) be a bounded sequence. Then there exists \( M > 0 \) such that \(|a_n| \leq M\) for all \( n = 1, 2, 3, \ldots \) This means
that the set $A = \{a_n \mid n \geq 1\}$ (the range of the sequence $a_n$) is bounded. If the set $A$ is finite, then $a_n$ must have a constant (and hence convergent) subsequence. If $A$ is infinite, then by Theorem 10, $A$ has a limit point, $a$. We claim that, in this case, $a$ is a cluster point of $a_n$. By Lemma 9, for every $\delta > 0$ the set $(a - \delta, a + \delta)$ contains infinitely many points of $A$. We will construct a subsequence of $a_n$ with limit $a$ by applying Lemma 9 for $\delta = 1$, $\delta = 1/2$, $\delta = 1/3$, etc. To this end, we note that there exists $n_1$ such that $a_{n_1} \in (a - 1, a + 1)$. Also, since $(a - \frac{1}{3}, a + \frac{1}{3})$ contains infinitely many points of $A$, there must exist $n_2 > n_1$ such that $a_{n_2} \in (a - \frac{1}{3}, a + \frac{1}{3})$. By similar reasoning, there exists $n_3 > n_2$ such that $a_{n_3} \in (a - \frac{1}{3}, a + \frac{1}{3})$, and so on. We have thus constructed a subsequence, $a_{n_k}$, of $a_n$ such that $|a_{n_k} - a| \leq 1/k$ for all $k = 1, 2, 3, \ldots$. It is clear that this subsequence has limit $a$. Thus, $a$ is a cluster point of $a_n$.

Now, assume that Theorem 17 is true and let $A$ be an infinite bounded subset of $\mathbb{R}$. Since $A$ is infinite, there exists a sequence $a_n$ such that $a_n \in A$ for all $n \geq 1$ and $a_i \neq a_j$ for all $i$ and $j$ with $i \neq j$. By Theorem 17, $a_n$ has a cluster point, $L$. This means that $a_n$ has a subsequence, $a_{n_k}$, with $\lim_{k \to \infty} a_{n_k} = L$. We claim that $L$ is a limit point of $A$. To see this, let $\delta > 0$ be given and consider the set $(L - \delta, L + \delta)$. Since $a_{n_k} \to L$, there exists a natural number $p$ such that $a_{n_k} \in (L - \delta, L + \delta)$ for all $k \geq p$. In particular, the set $(L - \delta, L + \delta)$ must contain infinitely many members of $A$. This shows, by Lemma 9, that $L$ is a limit point of $A$. $\blacksquare$

Now that we have shown that Theorems 10 and 17 are equivalent, we give the proof of Theorem 17. The proof rests on the following lemma.

**Lemma 18** Every sequence has a monotone subsequence.

**Proof.** Let $a_n$ be a sequence and suppose that $a_n$ has no monotone increasing subsequence. We will show that, in this case, $a_n$ must have a monotone decreasing subsequence.

The first step will be to show that for each $k \geq 1$, the set

$$A_k = \{a_n \mid n \geq k\}$$

must have a greatest member. With the goal of obtaining a contradiction, suppose that there exists $k \geq 1$ such that $A_k$ has no greatest member. Under this supposition, there must exist an integer $n_1 \geq 1$ such that $a_k < a_{k+n_1}$ (for otherwise, $a_k$ would be the greatest member of $A_k$). In addition, since
no member of
$$\{a_k, a_{k+1}, a_{k+2}, \ldots, a_{k+n}\}$$
can be the greatest member of $A_k$, then there exists an integer $n_2 > n_1$ such that $a_{k+n_2} < a_{k+n_2}$. Continuing this reasoning, we can obtain a sequence of positive integers $n_1, n_2, n_3, \ldots$ with $n_1 < n_2 < n_3 < \cdots$ such that $a_{k+n_1} < a_{k+n_2} < a_{k+n_3} < \cdots$, but this is a monotone increasing subsequence of $a_n$ and we are assuming that none exists. We conclude that for each $k \geq 1$, the set $A_k$ must have a greatest member and we are now prepared to construct a monotone decreasing subsequence of $a_n$. We do this as follows:

Since $A_1 = \{a_n \mid n \geq 1\}$ has a greatest member, there exists $n_1 \geq 1$ such that $a_n \leq a_{n_1}$ for all $n \geq 1$.

Since $A_{n_1+1} = \{a_n \mid n \geq n_1 + 1\}$ has a greatest member, there exists $n_2 \geq n_1 + 1$ such that $a_{n_1} \leq a_{n_2}$ for all $n \geq n_1 + 1$. Note that $a_{n_1} \geq a_{n_2}$.

Since $A_{n_2+1} = \{a_n \mid n \geq n_2 + 1\}$ has a greatest member, there exists $n_3 \geq n_2 + 1$ such that $a_{n_2} \leq a_{n_3}$ for all $n \geq n_2 + 1$. Note that $a_{n_3} \geq a_{n_3}$.

Continuing this reasoning, we obtain a monotone decreasing subsequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ of $a_n$. ■

Proof of BW Theorem -- Sequential Version. Let $a_n$ be a bounded sequence. By Lemma 18, there exists a monotone subsequence, $a_{n_k}$, of $a_n$. By Theorem 1.3.1 of the textbook, since $a_{n_k}$ is monotone and bounded, it converges. We conclude that $a_n$ has a cluster point. ■

Exercise 19

1. Prove that if $a_n$ is a non-convergent bounded sequence, then $a_n$ must have at least two distinct cluster points.

2. Prove or disprove: If $a_n$ is a sequence with cluster point $a$ and $b_n$ is a sequence with cluster point $b$, then $a + b$ is a cluster point of the sequence $a_n + b_n$.

3. Let $a_n$ be a bounded sequence and define

$$C = \{a \mid a \text{ is a cluster point of } a_n\}.$$

(a) Explain why sup $C$ and inf $C$ both exist.

(b) Prove that sup $C$ and inf $C$ are cluster points of $a_n$. (You will thus be proving that $C$ has a greatest and a least member.)

4. Give an example of an unbounded sequence that has no cluster points.

5. Give an example of an unbounded sequence that has cluster points.