Cauchy Sequences

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If \( a_n \) is a sequence that converges to the limit \( L \), then for any \( \varepsilon > 0 \), there exists an integer \( K \geq 1 \) such that \( |a_n - L| < \varepsilon/2 \) for all \( n \geq K \). Hence, if \( n \geq K \) and \( m \geq K \), we obtain

\[
|a_n - a_m| = |(a_n - L) - (a_m - L)| \\
\leq |a_n - L| + |a_m - L| \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
= \varepsilon.
\]

This shows that a convergent sequence must be a Cauchy sequence:

**Definition 1** A Cauchy sequence is a sequence, \( a_n \), such that for any \( \varepsilon > 0 \) there exists an integer \( K \geq 1 \) such that \( |a_n - a_m| < \varepsilon \) for all \( n \) and \( m \) with \( n \geq K \) and \( m \geq K \).

**Remark 2** If \( n = m \), then it is obvious that \( |a_n - a_m| = 0 < \varepsilon \) for any positive \( \varepsilon \), so in verifying that a sequence is a Cauchy sequence, it is only necessary to consider \( n \) and \( m \) with \( n \neq m \). If \( n \neq m \), then without loss of generality, we may assume that \( m > n \). Thus, an equivalent definition of Cauchy sequence is the following:

**Definition 3** A Cauchy sequence is a sequence, \( a_n \), such that for any \( \varepsilon > 0 \) there exists an integer \( K \geq 1 \) such that \( |a_n - a_m| < \varepsilon \) for all \( n \) and \( m \) with \( m > n \geq K \).

We have seen that every convergent sequence is a Cauchy sequence. The following theorem states that, in fact, the Cauchy property is both necessary and sufficient for convergence.
**Theorem 4** A sequence is convergent if and only if it is a Cauchy sequence.

**Proof.** We have already proved (in the introductory paragraph) that convergent sequences are Cauchy sequences. It remains to be proved that the converse is also true.

Let $a_n$ be a Cauchy sequence. Our strategy in showing that $a_n$ converges will be the following:

1. We will show that $a_n$ is bounded.

2. Knowing that $a_n$ is bounded, we will obtain a subsequence, $a_{n_k}$, of $a_n$ which converges to a limit $L$.

3. We will prove that the original sequence, $a_n$, must in fact converge to $L$.

Since $a_n$ is a Cauchy sequence, there exists an integer $K \geq 1$ such that $|a_n - a_m| < 1$ for all $n \geq K$ and $m \geq K$. In particular, this means that $|a_n - a_K| < 1$ for all $n \geq K$. We thus obtain

$$|a_n| = |(a_n - a_K) + a_K| \leq |a_n - a_K| + |a_K| < 1 + |a_K|$$

for all $n \geq K$. Hence if we let

$$M = \max \{1 + |a_K|, |a_1|, |a_2|, |a_3|, \ldots, |a_K|\},$$

then $|a_n| \leq M$ for all integers $n \geq 1$ and this shows that the sequence $a_n$ is bounded.

Since $a_n$ is bounded, it has a subsequence, $a_{n_k}$, that converges to a limit $L$. We will show that, in fact, $a_n$ converges to $L$.

Let $\varepsilon > 0$ be given.

Since $a_n$ is a Cauchy sequence, there exists an integer $p \geq 1$ such that $|a_n - a_m| < \varepsilon/2$ for all $n \geq p$ and $m \geq p$.

Since $\lim_{k \to \infty} a_{n_k} = L$, there exists an integer $q \geq p$ such that $|a_{n_q} - L| < \varepsilon/2$.

Hence, if $n \geq p$, then

$$|a_n - a_{n_q}| < \frac{\varepsilon}{2} \quad \text{(because } n \geq p \text{ and } n_q \geq q \geq p)$$
and this gives us
\[
|a_n - L| = |(a_n - a_m) - (a_m - L)| \\
\leq |a_n - a_m| + |a_m - L| \\
< \varepsilon + \varepsilon \\
= \varepsilon
\]
for all \( n \geq p \) and we have shown that \( \lim_{n \to \infty} a_n = L \). \( \blacksquare \)

One of the advantages of Theorem 4 is that we can use it to prove that a sequence converges even if we don’t have a candidate for the limit of the sequence. This is illustrated in the following example.

**Example 5** Consider the series

\[
\sum_{k=1}^{\infty} \frac{1}{k^3}.
\]

(1)

The sequence of partial sums of this series is

\[
s_1 = \frac{1}{1^2} \\
s_2 = \frac{1}{1^2} + \frac{1}{2^2} \\
s_3 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} \\
\vdots \\
s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^2} \\
\vdots
\]

We will show that \( s_n \) converges by showing that it is a Cauchy sequence.

Let \( \varepsilon > 0 \) be given. We must find an integer \( K \geq 1 \) such that \( |s_m - s_n| < \varepsilon \) for all \( n \) and \( m \) with \( m > n \geq K \). Let us “work backwards” to see how we can find such a \( K \). If \( m > n \geq 1 \), then \( m = n + j \) for some integer \( j \geq 1 \)
and we have

\[ |s_m - s_n| = |s_{n+j} - s_n| = \left| \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+j)^2} \right| \]

\[ = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+j)^2}. \]

We will now use induction on \( j \) to prove that

\[ \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+j)^2} \leq \frac{1}{n} - \frac{1}{n+j} \text{ for all } j \geq 1. \quad (2) \]

We remark that in proving that inequality (2) is true for all \( j \geq 1 \), we will actually be proving that the inequality is true for all \( n \geq 1 \) and for all \( j \geq 1 \) because \( n \) is assumed to be a fixed but arbitrary integer greater than or equal to 1.

For \( j = 1 \), the left hand side of inequality (2) is \( 1/(n+1)^2 \) and the right hand side is \( 1/n - 1/(n+1) \). Since

\[ \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \]

we see that inequality (2) is true for \( j = 1 \).

Assuming that inequality (2) is true for \( j = k \), i.e., assuming that

\[ \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+k)^2} \leq \frac{1}{n} - \frac{1}{n+k}. \quad (3) \]

we obtain

\[ \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+k)^2} + \frac{1}{(n+(k+1))^2} \leq \left( \frac{1}{n} - \frac{1}{n+k} \right) + \frac{1}{(n+k)(n+(k+1))} \]

\[ = \left( \frac{1}{n} - \frac{1}{n+k} \right) + \left( \frac{1}{n+k} - \frac{1}{n+(k+1)} \right) \]

\[ = \frac{1}{n} - \frac{1}{n+(k+1)} \]
which shows that the truth of inequality (2) for \( j = k \) implies its truth for \( j = k + 1 \). This completes our inductive proof that inequality (2) is true for all integers \( n \geq 1 \) and \( j \geq 1 \).

Since \( n \geq 1 \) and \( j \geq 1 \) implies that

\[
\frac{1}{n} - \frac{1}{n + j} < \frac{1}{n},
\]

we immediately conclude from inequality (2) that

\[
\frac{1}{(n + 1)^2} + \frac{1}{(n + 2)^2} + \cdots + \frac{1}{(n + j)^2} < \frac{1}{n}
\]

for all \( n \geq 1 \) and \( j \geq 1 \). This gives us

\[
|s_{n+j} - s_n| < \frac{1}{n} \text{ for all } n \geq 1 \text{ and } j \geq 1
\]

or equivalently

\[
|s_m - s_n| < \frac{1}{n} \text{ for all } m \text{ and } n \text{ with } m > n \geq 1.
\]

It is now clear that if we choose \( K > 1/\varepsilon \), then we will have \( |s_m - s_n| < \varepsilon \) for all \( m \) and \( n \) with \( m > n \geq K \). This shows that \( s_n \) is a Cauchy sequence and hence converges. By definition, this means that series (1) converges.

Note that we were able to prove that the series (1) converges without first coming up with a candidate for its sum (meaning the limit of its sequence of partial sums). If we had tried to prove convergence by using the definition of limit of a sequence, we would first have had to have come up with a candidate for what the limit is, but the limit is not obvious in this case. It is a famous result of Leonhard Euler (1707–1783) that

\[
\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.
\]

**Exercise 6** The sequence of partial sums, \( s_n \), of the series (1) is obviously monotone increasing. Hence, another way to show that this sequence converges is to show that it is bounded above (thus avoiding showing that it is a Cauchy sequence). Show that \( s_n \) is bounded above. (Hint: Use induction to show that \( s_n \leq 2 - 1/n \) for all \( n \geq 1 \).)
Exercise 7 Prove that the series

\[ \sum_{k=1}^{\infty} \frac{1}{k!} \]

converges. You can do this either by showing that the sequence of partial sums, \( s_n \), of this series is a Cauchy sequence or by showing that \( s_n \) is bounded above (since \( s_n \) is clearly monotone increasing).

If you choose to prove that \( s_n \) is a Cauchy sequence, the hint is to prove that

\[ |s_{n+j} - s_n| \leq \frac{1}{2^{n-1}} - \frac{1}{2^{n+(j-1)}} \quad \text{for all } n \geq 1 \text{ and } j \geq 1. \]

If you choose to prove that \( s_n \) is bounded above, then the hint is to prove that

\[ s_n \leq 2 - \frac{1}{2^{n-1}} \quad \text{for all } n \geq 1. \]

Whichever method you choose, a helpful fact is that \( 2^n \leq (n+1)! \) for all \( n \geq 1 \), so you should start by proving this first.

Exercise 8 In the textbook, Section 3.3, do problems 5, 7, 8, and 17.