Some Properties of Continuous Functions

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If \( f : A \rightarrow R \) and \( B \subseteq A \), then we say that \( f \) is continuous on \( B \) if \( f \) is continuous at each point of \( B \). If \( B = [a, b] \) (a closed interval) and \( f \) is continuous on \( B \), then a very important consequence of this is that \( f (B) \) is also a closed interval, i.e., \( f ([a, b]) = [c, d] \) for some \( c \) and \( d \). Before proving this fact, we look at a couple of examples that are familiar from Calculus.

**Example 1** The function \( f : R \rightarrow R \) defined by \( f (x) = x^2 \) is continuous on the interval \([-2, 5]\) and \( f ([−2, 5]) = [0, 25] \).

\[ \text{Graph of } f(x) = x^2 \text{ on } B = [-2, 5] \]

**Example 2** The function \( f : R \rightarrow R \) defined by \( f (x) = \sin x \) is continuous on the interval \([-3\pi/4, \pi/4]\) and \( f ([−3\pi/4, \pi/4]) = [−1, \sqrt{2}/2] \).
Example 3 Let $f : R \to R$ be the constant function $f(x) = 4$. Then $f$ is continuous on $[0, 1]$ and $f([0, 1]) = \{4\} = [4, 4]$.

Example 4 Let $f : R \to R$ be defined by $f(3) = 2$. Then $f$ is continuous on $[3, 3] = \{3\}$ and $f([3, 3]) = \{2\} = [2, 2]$.

Remark 5 Examples 3 and 4 are examples of extreme cases. The case of real interest is the case that $a < b$ and $f$ is not constant on $[a, b]$.

Exercise 6 If $f$ is not continuous on $[a, b]$, then it need not be true that $f([a, b])$ is a closed interval. Give an example of a function that is not continuous on a closed interval $[a, b]$ and for which $f([a, b])$ is not a closed interval. Also, give an example of a function that is not continuous on a closed interval $[a, b]$ and for which $f([a, b])$ is a closed interval.

We now state and prove our main result.

Theorem 7 Let $f : A \to R$ and let $a$ and $b$ be real numbers with $a < b$ such that the interval $[a, b]$ is contained in $A$ and such that $f$ is continuous and not constant on $[a, b]$. Then there exist real numbers $c$ and $d$ with $c < d$ such that $f([a, b]) = [c, d]$.

Proof. First, we prove that the set $f([a, b])$ is bounded above and below.

If $f([a, b])$ is not bounded above, then there is a sequence of points, $y_n$, in $f([a, b])$ such that $\lim_{n \to \infty} y_n = \infty$. For each $n$, there exists $x_n \in [a, b]$ such that $f(x_n) = y_n$. The sequence $x_n$ is bounded above (by $b$) and below (by $a$). In addition, $x_n$ must have a monotone subsequence, $x_{n_k}$, (by homework
problem 18 on page 81 of the textbook). This subsequence must converge to a point \( x_0 \in [a, b] \), i.e., \( \lim_{k \to \infty} x_{n_k} = x_0 \). Since \( f \) is continuous at \( x_0 \), we must have \( \lim_{k \to \infty} f (x_{n_k}) = \lim_{k \to \infty} y_{n_k} = f (x_0) \). Since \( f (x_0) \) is a real number, this contradicts \( \lim_{n \to \infty} y_n = \infty \). We conclude that \( f ([a, b]) \) must be bounded above. A similar argument shows that \( f ([a, b]) \) is also bounded below.

Since \( f ([a, b]) \) is non-empty and bounded above and below, then by the completeness axiom, \( f ([a, b]) \) must have a supremum and an infimum. Let \( c = \inf (f ([a, b])) \) and let \( d = \sup (f ([a, b])) \). Since \( f \) is not constant on \([a, b]\), it must be true that \( c < d \). We will show that \( f ([a, b]) = [c, d] \).

Since \( c \) is a lower bound and \( d \) is an upper bound of \( f ([a, b]) \), it is clear that \( f ([a, b]) \subseteq [c, d] \). Hence, we are left to show that \([c, d] \subseteq f ([a, b])\).

First, we show that \( c \in f ([a, b]) \) and \( d \in f ([a, b]) \). Since \( c = \inf (f ([a, b])) \), there is a sequence, \( y_n \), of points in \( f ([a, b]) \) such that \( y_n \to c \). There is a corresponding sequence, \( p_n \), of points in \([a, b]\) such that \( f (p_n) = y_n \). Just as in the first part of this proof, \( p_n \) must have a monotone subsequence which converges to a point \( p \in [a, b] \), and by continuity of \( f \) at \( p \), we obtain that \( f (p_n) \) converges to \( f (p) \) and hence that \( f (p) = c \). This shows that \( c \in f ([a, b]) \).

By a similar argument, we obtain a point \( q \in [a, b] \) such that \( f (q) = d \) and this shows that \( d \in f ([a, b]) \). Note that \( p \neq q \). We will assume for the rest of the proof that \( p < q \). As a homework exercise, show that the rest of the proof can be modified to accommodate the possibility that \( q < p \).

We now complete the proof by showing that if \( c < y_0 < d \), then \( y_0 \in f ([a, b]) \). Supposing that \( c < y_0 < d \), define

\[
K = \{ x \in [p, q] \mid f (x) < y_0 \}.
\]

Then \( K \neq \emptyset \) (because \( p \in K \) and \( K \) is bounded above (by \( q \)), so \( \sup (K) \) exists. Let \( x_0 = \sup (K) \). Then \( x_0 \leq q \) and there is a sequence, \( x_n \), of points in \( K \) such that \( x_n \to x_0 \). Since \( f \) is continuous at \( x_0 \), we must also have \( f (x_n) \to f (x_0) \). Moreover, since \( f (x_n) < y_0 \) for all \( n \), it must be the case that \( f (x_0) \leq y_0 < d \) and it follows from this that \( x_0 < q \). We will now show that \( f (x_0) = y_0 \).

If \( f (x_0) < y_0 \), then, since \( f \) is continuous at \( x_0 \) and since \( y_0 - f (x_0) > 0 \), there exists \( \delta > 0 \) such that \( |f (x) - f (x_0)| < y_0 - f (x_0) \) for all \( x \in [a, b] \) with \( |x - x_0| < \delta \). Furthermore, since \( x_0 < q \leq b \), the set \([a, b] \cap (x_0, x_0 + \delta)\) is non-empty. Let \( x^* \in [a, b] \cap (x_0, x_0 + \delta) \). Then \( |f (x^*) - f (x_0)| < y_0 - f (x_0) \) which means that \( f (x^*) < y_0 \) and hence that \( x^* \in K \). However \( x_0 < x^* \) so
this contradicts the fact that $x_0$ is an upper bound for $K$. We conclude that $f (x_0) = y_0$ and hence conclude that $y_0 \in f ([a, b])$. This completes the proof that $[c, d] \subseteq f ([a, b])$. ■

Two immediate corollaries of Theorem 7 are:

**Corollary 8 (Intermediate Value Theorem)** Let $f : A \to R$ and let $a$ and $b$ be real numbers with $a < b$ such that the interval $[a, b]$ is contained in $A$ and such that $f$ is continuous on $[a, b]$. Also, suppose that $f (a) < f (b)$ (or that $f (a) > f (b)$) and let $y_0$ be a number such that $f (a) < y_0 < f (b)$ (or $f (a) > y_0 > f (b)$). Then there exists $x_0 \in (a, b)$ such that $f (x_0) = y_0$.

**Corollary 9 (Extreme Value Theorem)** Let $f : A \to R$ and let $a$ and $b$ be real numbers with $a < b$ such that the interval $[a, b]$ is contained in $A$ and such that $f$ is continuous on $[a, b]$. Then $\max (f ([a, b]))$ and $\min (f ([a, b]))$ both exist. In other words, there exist points $p$ and $q \in [a, b]$ such that $f (p) \leq f (x)$ for all $x \in [a, b]$ and $f (q) \geq f (x)$ for all $x \in [a, b]$.

**Example 10** We can use the Intermediate Value Theorem to prove that if $f : [0, 1] \to R$ is continuous and $f ([0, 1]) \subseteq [0, 1]$, then $f$ must have a fixed point $x$, such that $f (x) = x$.

To prove that $f$ must have a fixed point, consider the function $g : [0, 1] \to R$ defined by $g (x) = f (x) - x$ and note that $g$ is continuous on $[0, 1]$. (Why?)

If $g (0) = 0$, then $f (0) = 0$ and hence 0 is a fixed point of $f$. If $g (1) = 0$, then $f (1) = 1$ and hence 1 is a fixed point of $f$. Thus, let us suppose that $g (0) \neq 0$ and $g (1) \neq 0$. In this case, we have

$$g (1) = f (1) - 1 < 0$$

and

$$g (0) = f (0) > 0$$

which gives us

$$g (1) < 0 < g (0).$$

By the Intermediate Value Theorem, there exists a point $x \in (0, 1)$ such that $g (x) = 0$ and for this point, $x$, we have $f (x) = x$, i.e., $x$ is a fixed point of $f$.

**Exercise 11** In the book, Section 2.6, do problems 21, 23, 25, and 27.