Continuity

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October 9, 2000

1 Definition and Examples

If \( f : A \rightarrow R \), we want to say that \( f \) is continuous at a point \( x_0 \in A \) if \( |f(x) - f(x_0)| \) is very small (as small as we like) for all points \( x \in A \) with \( |x - x_0| \) sufficiently small. This idea is captured in the following definition.

Definition 1 Let \( f : A \rightarrow R \) and let \( x_0 \in A \). We say that \( f \) is continuous at \( x_0 \) if for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( |f(x) - f(x_0)| < \epsilon \) for all \( x \in A \) with \( |x - x_0| < \delta \).

If \( f \) is continuous at each point in \( A \), then we say that \( f \) is continuous.

Example 2 Let us show that the function \( f : R \rightarrow R \) defined by \( f(x) = x^2 \) is continuous. To do this, we let \( x_0 \) be an arbitrary point in \( R \) and we show that \( f \) is continuous at \( x_0 \).

Let \( \epsilon > 0 \) be given and let \( \delta \) be a positive number such that \( \delta < \min \left\{ 1, \frac{\epsilon}{1 + 2|x_0|} \right\} \).

Then for all \( x \in R \) with \( |x - x_0| < \delta \), we have

\[
|x - x_0| < \frac{\epsilon}{1 + 2|x_0|}
\]

and we also have

\[
|x + x_0| = |(x - x_0) + 2x_0| \leq |x - x_0| + |2x_0| < 1 + |2x_0|
\]
which gives us

\[ |f(x) - f(x_0)| = |x^2 - x_0^2| = |x + x_0||x - x_0| < (1 + |2x_0|) \left( \frac{\epsilon}{1 + 2|x_0|} \right) = \epsilon \]

This shows that \( f \) is continuous at \( x_0 \). Since \( x_0 \) was chosen arbitrarily from \( R \), we conclude that \( f \) is continuous at all points of \( R \), i.e., \( f \) is continuous.

**Example 3** Let \( f : (R - \{0\}) \rightarrow R \) be the function defined by \( f(x) = 1/x^2 \). Let us show that \( f \) is continuous.

Let \( x_0 \in R - \{0\} \) and let \( \epsilon > 0 \) be given. We must show that there exists \( \delta > 0 \) such that \( |f(x) - f(x_0)| < \epsilon \) for all \( x \in R - \{0\} \) with \( |x - x_0| < \delta \).

We claim that if we take any \( \delta > 0 \) with

\[ \delta < \min \left\{ \frac{|x_0|}{2}, \frac{|x_0|^3 \epsilon}{10} \right\}, \]

then such a \( \delta \) will work. Indeed, if \( x \in R - \{0\} \) and \( |x - x_0| < \delta \), then we have

\[ |x - x_0| < \frac{|x_0|^3 \epsilon}{10} \]

and

\[ |x + x_0| = |(x - x_0) + 2x_0| \leq |x - x_0| + 2|x_0| < \frac{|x_0|}{2} + 2|x_0| = \frac{5|x_0|}{2} \]

and since

\[ ||x| - |x_0|| < |x - x_0| < \frac{|x_0|}{2} \]

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(by the triangle inequality), we also have

\[-\frac{|x_0|}{2} < |x| - |x_0|\]

\[\implies \frac{|x_0|}{2} < |x|\]

\[\implies \frac{x^2_0}{4} < x^2\]

\[\implies \frac{x^4_0}{4} < x^2 x^2_0\]

\[\implies 1 \cdot \frac{x^4}{x^4_0} < \frac{4}{x^4_0}\]

Thus, if \(x \in R - \{0\}\) and \(|x - x_0| < \delta\), we obtain

\[|f(x) - f(x_0)| = \left| \frac{1}{x^2} - \frac{1}{x^2_0} \right|\]

\[= \frac{1}{x^2 x^2_0} \cdot |x + x_0| \cdot |x - x_0|\]

\[< \frac{4}{x^4_0} \cdot \frac{5|x_0|}{2} \cdot \frac{|x_0|^3 \epsilon}{10}\]

\[= \epsilon\]

which shows that \(f\) is continuous at \(x_0\).

**Example 4** Consider the uninteresting function \(f : \{3\} \longrightarrow R\) defined by \(f(3) = 12\). Let us show that \(f\) is continuous.

Since the domain of \(f\) contains just one point \(3\), we must show that \(f\) is continuous at \(3\). To do this, we let \(\epsilon > 0\) be given and let \(\delta = 1\). Then, if \(x \in \{3\}\) and \(|x - 3| < \delta\), it must in fact be the case that \(x = 3\) and we have \(|f(x) - f(3)| = 0 < \epsilon\). This shows that \(f\) is continuous.

**Exercise 5** Let \(f : R \longrightarrow R\) be defined by \(f(x) = 3x\). Show that \(f\) is continuous.

**Exercise 6** Let \(f : R \longrightarrow R\) be defined by

\[f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 12 & \text{if } x = 0 \end{cases}\]

Show that \(f\) is continuous at all points in \(R\) except at the point \(x_0 = 0\).
Exercise 7 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that $f$ is continuous at $x_0 = 0$.

Exercise 8 Let $f : \{2, 2.07\} \rightarrow \mathbb{R}$ be the function defined by $f(2) = 3$, $f(2.07) = 6$. Show that $f$ is continuous.

Exercise 9 Let $f$ be a sequence. Show that $f$ is continuous.

The above examples and exercises illustrate that checking the continuity of a function $f$ at a point $x_0$ boils down to two cases: the case that $x_0$ is a limit point of the domain of $f$ and the case that $x_0$ is an isolated point of the domain of $f$. Recall that a limit point of a set $A$ is a point $x_0$ (not necessarily in $A$) such that for every $\delta > 0$, the set $(x_0 - \delta, x_0 + \delta) \cap (x_0, x_0 + \delta)$ contains points of $A$. An isolated point of $A$ is defined to be a point in $A$ that is not a limit point of $A$. For example, the set $N$ (the set of all natural numbers) consists entirely of isolated points.

If $A$ is any non-empty set, then every member of $A$ is either a limit point of $A$ or an isolated point of $A$. If $f : A \rightarrow \mathbb{R}$ and $x_0$ is an isolated point of $A$, then it is automatic that $f$ is continuous at $x_0$. Indeed, since $x_0$ is not a limit point of $A$, there exists $\delta > 0$ such that $x_0$ is the only member of $A$ that lies in the interval $(x_0 - \delta, x_0 + \delta)$, i.e. $x_0$ is the only member, $x$, of $A$ that satisfies $|x - x_0| < \delta$. Hence, for any given $\epsilon > 0$, we can use the $\delta$ just defined to show that $f$ is continuous at $x_0$.

On the other hand, if $f : A \rightarrow \mathbb{R}$ and $x_0$ is a member of $A$ which is a limit point of $A$, then continuity of $f$ at $x_0$ is equivalent $\lim_{x \to x_0} f(x) = f(x_0)$. In summary, if $f : A \rightarrow \mathbb{R}$ and $x_0 \in A$, then:

1. If $x_0$ is an isolated point of $A$, then $f$ is continuous at $x_0$.
2. If $x_0$ is a limit point of $A$, then $f$ is continuous at $x_0$ if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

2 Continuity and Sequences

The following theorem and its corollary point out important connections between limits of functions at a point and limits of associated sequences.
Theorem 10 Suppose that $f : A \to R$, $x_0$ is a limit point of $A$, and $L$ is a real number. Then the following two statements are equivalent.

1. $\lim_{x \to x_0} f(x) = L$.

2. If $x_n$ is a sequence of points in $A - \{x_0\}$ with $x_n \to x_0$, then $f(x_n) \to L$.

Proof. First we show that the truth of statement 1 implies the truth of statement 2.

Suppose that $\lim_{x \to x_0} f(x) = L$ and let $x_n$ be a sequence of points in $A - \{x_0\}$ with $x_n \to x_0$.

Let $\varepsilon > 0$ be given.

Since $\lim_{x \to x_0} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in A$ with $0 < |x - x_0| < \delta$.

Since $x_n \to x_0$, there exists an integer $M \geq 1$ such that $|x_n - x_0| < \delta$ for all $n \geq M$.

Hence, $|f(x_n) - L| < \varepsilon$ for all $n \geq M$ and we have shown that $f(x_n) \to L$.

Now we show that the truth of statement 2 implies the truth of statement 1. We will do this by proving the contrapositive; i.e., we will show that if statement 1 is false, then statement 2 must be false.

Suppose that it is not true that $\lim_{x \to x_0} f(x) = L$.

Then there exists $\varepsilon > 0$ such and a point $x_1 \in A$ with $0 < |x_1 - x_0| < 1$ and $|f(x_1) - L| \geq \varepsilon$. By the same token, there exists a point $x_2 \in A$ with $0 < |x_2 - x_0| < 1/2$ and $|f(x_2) - L| \geq \varepsilon$. Continuing inductively, we see that for any positive integer, $n$, there exists a point $x_n \in A$ such that $0 < |x_n - x_0| < 1/n$ and $|f(x_n) - L| \geq \varepsilon$. The sequence, $x_n$, thus constructed is a sequence of points in $A - \{x_0\}$ that converges to $x_0$ but the corresponding sequence, $f(x_n)$, does not converge to $L$. ■

Corollary 11 Suppose that $f : A \to R$ and suppose that $x_0 \in A$. Then the following statements are equivalent:

1. $f$ is continuous at $x_0$.

2. If $x_n$ is a sequence of points in $A - \{x_0\}$ with $x_n \to x_0$, then $f(x_n) \to f(x_0)$.
Proof. If $x_0$ is a limit point of $A$, then equivalence of 1 and 2 follows immediately from Theorem 10. If $x_0$ is an isolated point of $A$, then both statements 1 and 2 are true and hence equivalent. ■

Remark 12 Corollary 11 remains true if $A - \{x_0\}$ in statement 2 is replaced with $A$. The proof in this case requires only slight modification. However, in statement 2 of Theorem 10, $A - \{x_0\}$ cannot be replaced with $A$. (Make sure you understand why.)

Example 13 Let $f : R \to R$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$ 

We can use Corollary 11 to show that $f$ is not continuous at any point $x_0 \in R$. This is done by considering two cases: the case that $x_0$ is rational and the case that $x_0$ is irrational.

Suppose that $x_0$ is rational. Then $f(x_0) = 1$. Also, since every interval must contain irrational numbers (earlier homework exercise), there exists a sequence, $x_n$, of irrational numbers such that $x_n \to x_0$. Since $f(x_n) = 0$ for all $n$, we have $f(x_n) \to 0$; i.e., $f(x_n)$ does not have limit $f(x_0)$. By Corollary 11, we conclude that $f$ is not continuous at $x_0$.

Consideration of the case that $x_0$ is irrational is similar and is left as homework.

Example 14 The “ruler function” is the function $f : R \to R$ defined as follows:

1. $f(0) = 1$
2. If $x$ is a non-zero rational number, then write $x$ as $x = m/n$ where $m$ and $n$ are integers with no common factors and $n > 0$, and define $f(x) = 1/n$.
3. If $x$ is irrational, then $f(x) = 0$. 

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For example,

\[
f(2) = f \left( \frac{2}{1} \right) = \frac{1}{1} = 1
\]

\[
f \left( \frac{3}{4} \right) = \frac{1}{4}
\]

\[
f \left( -\frac{13}{18} \right) = \frac{1}{18}
\]

\[
f \left( \sqrt{2} \right) = 0
\]

A fascinating animated graph of this function can be seen at Dr. Jonathan Lewin's Web site http://science.kennesaw.edu/~jlewin/ruler-function/.

We can use Corollary 11 to show that if \( x_0 \) is rational, then \( f \) is not continuous at \( x_0 \). If \( x_0 \) is rational, then just as in the previous example, we take a sequence of irrational numbers, \( x_n \), converging to \( x_0 \). For this sequence, we have \( f(x_n) = 0 \) for all \( n \) so \( f(x_n) \to 0 \). However \( f(x_0) \neq 0 \), so \( f \) is not continuous at \( x_0 \).

**Exercise 15** Show that if \( x_0 \) is irrational, then the ruler function (preceding example) is continuous at \( x_0 \).

**Exercise 16** Suppose that \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( f(x+y) = f(x) + f(y) \) for all \( x \) and \( y \in \mathbb{R} \). (Such a function is said to be “additive”.) Prove the following:

1. \( f(0) = 0 \)
2. If \( r \) is any rational number, then \( f(r) = r \cdot f(1) \).
3. If \( \lim_{x \to 0} f(x) \) exists, then \( f \) is continuous at 0. (HINT: Use Theorem 10 and parts 1 and 2 above.)
4. If \( \lim_{x \to 0} f(x) \) exists, then \( f \) is continuous at every point in \( \mathbb{R} \).
5. If \( \lim_{x \to 0} f(x) \) exists, then there is a constant, \( c \), such that \( f(x) = cx \) for all \( x \in \mathbb{R} \). (HINT: The constant is \( c = f(1) \). Let \( x_0 \) be any real number and take a sequence of rational numbers, \( r_n \), with \( r_n \to x_0 \). Then use the results of parts 2 and 4 above.)

**Exercise 17** In the book, Section 2.5, do problems 4, 5, and 19.