Basic Facts About Limits of Sequences

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1 Some Other Definitions of Limit

If \( f : N \to R \) is a sequence defined by \( f(n) = a_n \) and if \( L \) is a real number, the
definition of \( \lim_{n \to \infty} f(n) = L \) that we have given is the following:

**Definition 1** We say that the sequence \( f \) has limit \( L \) and we write \( \lim_{n \to \infty} f(n) = L \) (or simply \( a_n \to L \)) if for every \( \epsilon > 0 \), there exists \( n_0 \in N \) such that
\[ |a_n - L| \leq \epsilon \] for all \( n \in N \) with \( n \geq n_0 \).

The following definition is equivalent.

**Definition 2** We say that the sequence \( f \) has limit \( L \) and we write \( \lim_{n \to \infty} f(n) = L \) (or simply \( a_n \to L \)) if for every \( \epsilon > 0 \), there exists \( n_0 \in N \) such that
\[ |a_n - L| < \epsilon \] for all \( n \in N \) with \( n \geq n_0 \).

Definition 2 at first appears to be different from Definition 1 because Definition 1 requires that there exist \( n_0 \) such that \( |a_n - L| \leq \epsilon \) for all \( n \geq n_0 \), whereas, Definition 2 has the seemingly stricter requirement that there exist \( n_0 \) such that \( |a_n - L| < \epsilon \) for all \( n \geq n_0 \). However, the requirement of Definition 2 is really not stricter because if we are given an \( \epsilon > 0 \), then an \( n_0 \) that satisfies \( |a_n - L| \leq \epsilon/2 \) for all \( n \geq n_0 \) will satisfy \( |a_n - L| < \epsilon \) for all \( n \geq n_0 \). In other words, for a given \( \epsilon > 0 \), the corresponding \( n_0 \) for Definition 2 might need to be chosen larger than would be needed to satisfy Definition 1 but such an \( n_0 \) is guaranteed to exist (assuming that \( a_n \to L \)).

Another equivalent definition of \( \lim_{n \to \infty} a_n = L \) is the following:

**Definition 3** We say that the sequence \( f \) has limit \( L \) and we write \( \lim_{n \to \infty} f(n) = L \) (or simply \( a_n \to L \)) if for every \( \epsilon > 0 \), there exists \( n_0 \in N \) such that
\( f(N \cap [n_0, \infty)) \subseteq (L - \epsilon, L + \epsilon) \).

To see that Definition 3 is equivalent to Definition 2, note that
\[ N \cap [n_0, \infty) = \{n_0, n_0 + 1, n_0 + 2, \ldots \} \]
\[ f(N \cap [n_0, \infty)) = \{a_n \mid n \geq n_0 \} \]
and

$$(L - \epsilon, L + \epsilon) = \{x \in R \mid |x - L| < \epsilon\}.$$ 

Using Definition 3, it is easy to see that $\lim_{n \to \infty} \left((1 + (-1)^n)\right)$ does not exist. Given any $L \in R$, we can choose $\epsilon > 0$ so that either $0 \notin (L - \epsilon, L + \epsilon)$ or $2 \notin (L - \epsilon, L + \epsilon)$. (In fact $\epsilon = 1/2$ will work no matter what $L$ is.) Having chosen this $\epsilon$, we note that for any $n_0 \in N$ we have $f(N \cap [n_0, \infty)) = \{0, 2\}$. Thus, no matter how large we choose $n_0$, will can never achieve $f(N \cap [n_0, \infty)) \subseteq (L - \epsilon, L + \epsilon)$.

There are other small changes that can be made to obtain other equivalent definitions of limit. For example, in Definition 3, we could use the interval $[L - \epsilon, L + \epsilon]$, in place of the interval $(L - \epsilon, L + \epsilon)$.

## 2 Basic Limit Theorems

One of the most basic facts about limits is that the limit of a convergent sequence is unique; that is, if $\lim_{n \to \infty} f(n) = A$ and $\lim_{n \to \infty} f(n) = B$, then it must be the case that $A = B$. (A proof of this is given in the textbook. Try to prove it yourself before looking there.) Another rather intuitive result is the so-called *Squeeze Theorem* or *Sandwich Theorem* which says that if $f$, $g$, and $h$ are sequences such that $f(n) \leq g(n) \leq h(n)$ for all $n$ and if $f(n) \to L$ and $h(n) \to L$, then it must also be the case that $g(n) \to L$. The proof of this theorem is also given in the book. In what follows, we examine some other important properties of convergent sequences.

A sequence, $f$, is said to be *bounded* if its range is bounded. In other words, $f$ is said to be bounded if $f(N)$ is a bounded subset of $R$. Our first theorem about limits is that a convergent sequence must be bounded.

**Theorem 4** If $f$ is a convergent sequence, then $f$ is bounded.

**Proof.** Suppose that $f$ is a sequence that converges to the limit $L \in R$. Then there exists $n_0 \in N$ such that $f(N \cap [n_0, \infty)) \subseteq (L - 1, L + 1)$. Since the set $(L - 1, L + 1)$ is bounded, the set $f(N \cap [n_0, \infty))$ is also bounded (because subsets of bounded sets are bounded).

Also, the set $f(\{1, 2, \ldots, n_0\})$ is finite, so it is bounded.

Since $f(N) = f(\{1, 2, \ldots, n_0\}) \cup f(N \cap [n_0, \infty))$, we conclude that $f(N)$ is bounded (because a union of bounded sets is bounded). $\blacksquare$

**Exercise 5** The above proof relies on the fact that a subset of a bounded set is bounded and also relies on the fact that a union of bounded sets is bounded. Prove these facts.

Our next theorem states that a sequence, $f$, converges to a limit, $L$, if and only if every subsequence of $f$ converges to $L$. Before we state this theorem, we need to define what we mean by a subsequence of $f$. We first provide a motivational discussion.
Let $f$ be a sequence. To construct a subsequence, $g$, of $f$, we let $n_1$ be a positive integer and define $g(1) = f(n_1)$. Next, we let $n_2$ be an integer with $n_2 > n_1$ and define $g(2) = f(n_2)$. Continuing in this fashion, for each $k > 1$, we choose an integer $n_k$ with $n_k > n_{k-1}$ and we define $g(k) = f(n_k)$. The sequence, $g$, thus obtained is a sequence whose terms have been plucked from among the terms of $f$ as the terms of $f$ are traversed from left to right.

For example, suppose that $f$ is the sequence $f(n) = 2n$. The terms of $f$ are

$$2, 4, 6, 8, 10, \ldots$$

A subsequence, $g$, of $f$ is the sequence with terms

$$2, 4, 8, 16, 32, \ldots$$

(i.e., $g$ is the sequence whose terms are powers of 2). Referring to the notation used in the discussion in the above paragraph, we see that to construct $g$, we use $n_1 = 1$, $n_2 = 2$, $n_3 = 4$, $n_4 = 8$, $n_5 = 16$, and in general $n_k = 2^{k-1}$. Thus $g(k) = f(n_k) = 2n_k = 2 \cdot 2^{k-1} = 2^k$ for all integers $k \geq 1$.

Another example of a subsequence of $f$ is the sequence, $h$ with terms

$$10, 100, 1000, \ldots$$

The sequence $h$ is defined by $h(k) = 10^k$.

**Exercise 6** Referring to the discussion above, find the increasing sequence of integers, $n_k$, such that $h(k) = f(n_k)$ for all integers $k \geq 1$.

We now give our formal definition of a subsequence.

**Definition 7** Let $f$ be a sequence and let $n_1, n_2, n_3, \ldots$ be positive integers such that $n_1 < n_2 < n_3 < \ldots$. The sequence, $g$, defined by $g(k) = f(n_k)$ for all integers $k \geq 1$ is called a subsequence of $f$.

**Exercise 8** Prove that if $n_1, n_2, n_3, \ldots$ are positive integers such that $n_1 < n_2 < n_3 < \ldots$, then $n_k \geq k$ for all $k \geq 1$. (Hint: Use proof by induction.)

**Exercise 9** Prove that if $f$ is a sequence, then $f$ is a subsequence of $f$.

We now give the main theorem concerning subsequences.

**Theorem 10** A sequence, $f$, converges to a limit, $L$, if and only if every subsequence of $f$ converges to $L$.

**Proof.** Suppose that $f$ converges to $L$ and let $g$ be a subsequence of $f$. Then there is a sequence of positive integers, $n_k$, such that $n_k < n_{k+1}$ and $g(k) = f(n_k)$ for all integers $k \geq 1$. With the goal of proving that $g$ converges to $L$, we let $\epsilon > 0$ be given. Since $f$ converges to $L$, there exists a positive integer $M$ such that $|f(n) - L| \leq \epsilon$ for all $n \geq M$. However, if $k \geq M$, then $n_k \geq M$.
by the result of Exercise 8, which means that \(|g(k) - L| = |f(n_k) - L| \leq \epsilon\) for all \(k \geq M\). This shows that \(g\) converges to \(L\).

The proof of the converse is easy. If every subsequence of \(f\) converges to \(L\), then since \(f\) is a subsequence of \(f\), it clearly must be the case that \(f\) converges to \(L\). ■

The next theorem addresses convergence of scalar multiples, sums and products of convergent sequences. If \(f\) and \(g\) are sequences defined by \(f(n) = a_n\) and \(g(n) = b_n\), then the sum, \(f + g\), is the sequence defined by \((f + g)(n) = a_n + b_n\) and the product, \(fg\), is the sequence defined by \((fg)(n) = a_n \cdot b_n\). Also, if \(g(n) \neq 0\) for any \(n\), then the reciprocal, \(1/g\), is the sequence defined by \((1/g)(n) = 1/g(n)\). More generally, the quotient, \(f/g\), is the sequence defined by \((f/g)(n) = f(n)/g(n)\). If \(k\) is a constant, then the scalar multiple, \(kf\), is the sequence defined by \((kf)(n) = k \cdot f(n)\)

**Theorem 11** Let \(f\) be a sequence which converges to \(A\) and let \(g\) be a sequence which converges to \(B\). (Where we discuss reciprocals and quotients, we also require that \(g(n) \neq 0\) for any \(n\) and that \(B \neq 0\)). Then:

1. If \(k\) is constant, then \(kf\) converges to \(kA\).
2. \(f + g\) converges to \(A + B\).
3. \(fg\) converges to \(AB\).
4. \(1/g\) converges to \(1/B\).
5. \(f/g\) converges to \(A/B\).

**Proof.** We prove 2 and 4 and leave the remaining proofs as homework. Throughout, we assume that \(f(n) = a_n\) and \(g(n) = b_n\) for all \(n \geq 1\).

**Proof of 2:** Let \(\epsilon > 0\) be given. Since \(f\) converges to \(A\), there exists an integer \(M_1\) such that \(|a_n - A| < \epsilon/2\) for all \(n \geq M_1\). Likewise, since \(g\) converges to \(B\), there exists an integer \(M_2\) such that \(|b_n - B| < \epsilon/2\) for all \(n \geq M_2\). If we let \(M = \max\{M_1, M_2\}\), then we have both \(|a_n - A| < \epsilon/2\) and \(|b_n - B| < \epsilon/2\) for all \(n \geq M\).

Now, note that

\[
|f(n) + g(n) - (A + B)| = |(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B|
\]

by the triangle inequality. From this, we obtain

\[
|(a_n + b_n) - (A + B)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

for all \(n \geq M\), which proves that \(\lim_{n \to \infty} (a_n + b_n) = A + B\).

**Proof of 4:** Let \(\epsilon > 0\) be given. Since \(g\) converges to \(B\) and \(B > 0\) (by assumption), there exists an integer \(M_1\) such that \(b_n \geq B/2\) for all \(n \geq M_1\). Note that this implies that \(Bb_n \geq B^2/2\) for all \(n \geq M_1\). Also, since \(g\) converges
to $B$, there exists an integer $M_2$ such that $|b_n - B| \leq B^2 \epsilon / 2$ for all $n \geq M_2$.
Let $M = \max \{M_1, M_2\}$. Then for all $n \geq M$, we have

$$\left| \frac{1}{b_n} - \frac{1}{B} \right| = \left| \frac{B - b_n}{b_n B} \right|$$

$$= \frac{|b_n - B|}{b_n B}$$

$$= |b_n - B| \cdot \frac{1}{B b_n}$$

$$\leq \frac{B^2 \epsilon}{2} \cdot \frac{2}{B^2}$$

$$= \epsilon$$

This proves that $1/b$ converges to $1/B$. ■

**Exercise 12** In the textbook, Section 1.2, do problems 1, 2, 8, 10, 13, 14 and 31.