Monotone Sequences

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Definition 1 A sequence, \( f \), defined by \( f(n) = a_n \), is said to be monotone increasing if \( a_n \leq a_{n+1} \) for all \( n \geq 1 \). Likewise, \( a_n \) is said to be monotone decreasing if \( a_n \geq a_{n+1} \) for all \( n \geq 1 \). If \( f \) is either monotone increasing or monotone decreasing, then we say that \( f \) is monotone.

Example 2 It is easy to see that the sequence \( a_n = n^2 \) is monotone increasing and that the sequence \( a_n = 1/n^2 \) is monotone decreasing.

Example 3 The sequence \( a_n = 1 + (-1)^n \) is obviously not monotone.

If a sequence, \( f(n) = a_n \), is monotone increasing and not bounded above, then it must be the case that \( a_n \to \infty \). To see this, note that if \( M \) is any real number, then \( M \) is not an upper bound for the range of \( f \). This means that there exists \( n_0 \) such that \( a_{n_0} > M \). Since \( f \) is monotone increasing, it must then be the case that \( a_n > M \) for all \( n \geq n_0 \) and this shows that \( a_n \to \infty \) (because \( M \) was arbitrary). By similar reasoning, we can conclude that if \( f \) is monotone decreasing and not bounded below, then \( a_n \to -\infty \).

We now ask what happens in the case that \( f \) is a monotone increasing (decreasing) sequence that is bounded above (below). Must \( f \) converge in these cases? The answer, which is “Yes”, is a consequence of the Completeness Axiom.

Theorem 4 If \( f \) is a monotone increasing sequence which is bounded above, then \( f \) converges. Likewise, if \( f \) is a monotone decreasing sequence which is bounded below, then \( f \) converges.

Furthermore, in the former case, \( f \) converges to the supremum of its range, i.e., \( \lim_{n \to \infty} f(n) = \sup(f(N)) \), and in the latter case, \( f \) converges to the infimum of its range, i.e. \( \lim_{n \to \infty} f(n) = \inf(f(N)) \).
Proof. We prove that if $f$ is monotone increasing and bounded above, then $f$ converges to $\sup (f(N))$. The proof of the other claim is similar and is left as an exercise. Throughout the proof, we assume that $f(n) = a_n$.

Since the range of $f$, $f(N) = \{a_n | n \geq 1\}$, is non-empty and bounded above (and since $R$ is complete), we know that $\sup (f(N))$ exists. Defining $L = \sup (f(N))$, we want to show that $\lim_{n \to \infty} f(n) = L$.

Let $\epsilon > 0$ be given. Then, since $L = \sup (f(N))$, we know (by Theorem 0.4.2 of the textbook) that some member of $f(N)$ must lie between $L - \epsilon$ and $L$, i.e. we know that there exists $n_0$ such that $L - \epsilon < a_{n_0} \leq L$. Furthermore, since $f$ is monotone increasing, we can conclude that $L - \epsilon < a_n$ for all $n \geq n_0$ and, since $L$ is an upper bound for $f(N)$, we know that $a_n \leq L$ for all $n$. This gives us $L - \epsilon < a_n < L + \epsilon$ for all $n \geq n_0$, or equivalently, $|a_n - L| < \epsilon$ for all $n \geq n_0$. We conclude that $\lim_{n \to \infty} f(n) = L$. □

Example 5 The sequence $f(n) = 1/n^2$ is monotone decreasing and bounded below. By Theorem 4, we can immediately conclude that $f$ converges. Also, it is easy to verify that $\inf (f(N)) = 0$, so we conclude that $\lim_{n \to \infty} f(n) = 0$.

Since no finite number of terms of a sequence can have an effect on whether or not the sequence converges, Theorem 4 is easily generalized to include sequences which are not monotone but which are eventually monotone as defined below.

Definition 6 A sequence, $f$, defined by $f(n) = a_n$, is said to be eventually monotone increasing if there exists an integer $k \geq 1$ such that $a_n \leq a_{n+1}$ for all $n \geq k$. Likewise, $a_n$ is said to be eventually monotone decreasing if there exists an integer $k \geq 1$ such that $a_n \geq a_{n+1}$ for all $n \geq k$. If $f$ is either eventually monotone increasing or eventually monotone decreasing, then we say that $f$ is eventually monotone.

A corollary to Theorem 4 is:

Corollary 7 If $f$ is an eventually monotone increasing sequence which is bounded above, then $f$ converges. Likewise, if $f$ is an eventually monotone decreasing sequence which is bounded below, then $f$ converges.

Remark 8 Note that Corollary 7 does not make that additional claim (as in Theorem 4) that $\lim_{n \to \infty} f(n) = \sup (f(n))$ (or $\inf (f(N))$). It is easy
to see that this additional claim might not be true in the case of an eventually monotone sequence. For example, consider the sequence, \( f \), obtained by inserting \(-1\) as the first term in the sequence \(1/n\). That is, let \( f \) be the sequence with terms

\[
-1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots
\]

This sequence is eventually monotone decreasing because \( f(n) \geq f(n+1) \) for all \( n \geq 2 \) and it is clear that this sequence has limit 0. However, note that \( \inf(f(N)) = -1 \) so it is not true that \( \lim_{n \to \infty} f(n) = \inf(f(N)) \).

The following example shows how elementary calculus tools (in particular, the derivative) can be used to show that a given sequence is eventually monotone. In some sense, it is illegal for us to use the derivative (yet) in this course because the point of the course is, in fact, to develop calculus from the ground up. However, let us take the slightly different attitude that we believe what we learned in earlier calculus courses is all correct and true and we are now in the process of going back and studying the fine details.

**Example 9** We will show that the sequence \( a_n = 1/(4n^2 - 80n + 401) \) is eventually monotone decreasing. First, let’s look at the first fifteen terms of this sequence (shown in the following table).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a_n ) (approximate)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.003</td>
</tr>
<tr>
<td>2</td>
<td>0.004</td>
</tr>
<tr>
<td>3</td>
<td>0.005</td>
</tr>
<tr>
<td>4</td>
<td>0.007</td>
</tr>
<tr>
<td>5</td>
<td>0.010</td>
</tr>
<tr>
<td>6</td>
<td>0.015</td>
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<tr>
<td>7</td>
<td>0.027</td>
</tr>
<tr>
<td>8</td>
<td>0.059</td>
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<td>9</td>
<td>0.200</td>
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<tr>
<td>10</td>
<td>1.000</td>
</tr>
<tr>
<td>11</td>
<td>0.200</td>
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<tr>
<td>12</td>
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</tr>
<tr>
<td>13</td>
<td>0.027</td>
</tr>
<tr>
<td>14</td>
<td>0.015</td>
</tr>
<tr>
<td>15</td>
<td>0.010</td>
</tr>
</tbody>
</table>
We can see that $a_n$ is actually increasing for $1 \leq n \leq 10$ but then $a_n$ starts to decrease. We will show that $a_n \geq a_{n+1}$ for all $n \geq 10$. First note that

$$4n^2 - 80n + 401 = 4(n - 10)^2 + 1$$

which shows that $a_n > 0$ for all $n$. If we can show that $g(n) = 4(n - 10)^2 + 1$ is increasing for $n \geq 10$, then this will be sufficient to conclude that $a_n = 1/g(n)$ is decreasing for $n \geq 10$. If we extend the domain of definition of $g$ to include all real numbers, i.e., think of $g$ as $g(x) = 4(x - 10)^2 + 1$ defined for all real numbers $x$, then $g'(x) = 8(x - 10)$ which is positive for all $x > 10$. This shows that $g$ is increasing on the interval $(10, \infty)$ which means, in particular, that $g(n) \leq g(n + 1)$ for all integers $n \geq 10$. We conclude that $a_n \geq a_{n+1}$ for all $n \geq 10$.

\begin{center}
Graph of $f(x) = \frac{1}{4(x-10)^2 + 1}$
\end{center}

**Remark 10** The preceding example used the following type of reasoning: If $f : [1, \infty) \rightarrow R$ is a function which is monotone decreasing on some interval $[k, \infty)$ where $k$ is an integer, then the sequence obtained by restricting the domain of $f$ to $N$ is eventually monotone decreasing. In particular, we can conclude that $f(n) \geq f(n + 1)$ for all integers $n \geq k$. However, the converse of this fact is not true. If $f : [1, \infty) \rightarrow R$ and if we know that the sequence $f(1), f(2), f(3), \ldots$ is eventually decreasing (say for $n \geq k$), we may not, in general, conclude that the function $f$ is decreasing on the interval $[k, \infty)$ (or, for that matter, on any interval of the form $[b, \infty)$). As an example, consider the function $f : [1, \infty) \rightarrow R$ defined by

$$f(x) = \sin(2\pi x) + \frac{1}{x}.$$
If we restrict the domain of $f$ to $N$, we obtain the sequence $f(n) = 1/n$ (because $\sin (2\pi n) = 0$ for all integers $n$). This sequence is clearly monotone decreasing, but the function $f$ is not monotone decreasing on any interval of the form $[a, \infty)$.

![Graph of $f(x) = \sin (2\pi x) + 1/x$]

**1 Proof by Induction**

If we want to prove that a statement, $P(n)$, is true for all integers $n \geq 1$, then we can attempt to do this by using the principle of *mathematical induction*. The principle of induction is:

If $P(1)$ is true and if the truth of $P(k)$ implies the truth of $P(k + 1)$, then $P(n)$ is true for all $n \geq 1$.

The reasoning behind the principle of induction is that if $P(1)$ is true and if the truth of $P(k)$ implies the truth of $P(k + 1)$, then since $P(1)$ is true and the truth of $P(1)$ implies the truth of $P(2)$, we know that $P(2)$ is true. Since $P(2)$ is true and the truth of $P(2)$ implies the truth of $P(3)$, then we know that $P(3)$ is true, and so on.

We see then that giving a proof by induction involves two things:

1. showing that $P(1)$ is true

2. showing that if $P(k)$ is true (where $k$ is assumed to be an arbitrary positive integer), then $P(k + 1)$ must be true.
Example 11 As an example of a proof by induction, we prove that if \( n \) is any positive integer, then

\[
1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}. \tag{1}
\]

(You may remember using formula (1) in Calculus in computing certain integrals.)

The statement, \( P(n) \), that we want to prove for all \( n \geq 1 \) is the statement given by formula (1). \( P(1) \) is the statement

\[
1 = \frac{1(1+1)}{2}
\]

which is obviously true.

For any integer \( k \geq 1 \), \( P(k) \) is the statement

\[
1 + 2 + 3 + \ldots + k = \frac{k(k + 1)}{2}
\]

and \( P(k + 1) \) is the statement

\[
1 + 2 + 3 + \ldots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}.
\]

We need to show that \( P(k + 1) \) is true under the assumption that \( P(k) \) is true.

Assuming that \( P(k) \) is true, we have

\[
1 + 2 + 3 + \ldots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1)
= \frac{k(k + 1)}{2} + \frac{2(k + 1)}{2}
= \frac{(k + 1)(k + 2)}{2}
\]

and hence, the truth of \( P(k) \) implies the truth of \( P(k + 1) \).

This proof by induction shows that \( P(n) \) is true for all integers \( n \geq 1 \).

Exercise 12 Use proof by induction to prove that the following summation formulas (both of which you might also remember from Calculus) are true for all integers \( n \geq 1 \):

\[
1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}
\]
\[1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.\]

Proof by induction is often useful in showing that a given sequence is monotone and/or bounded. This is illustrated in the following example.

**Example 13** Consider the sequence defined by \(a_1 = \sqrt{2}, \ a_{n+1} = \sqrt{2} + a_n\)
for all \(n \geq 1\). We will show that this sequence is monotone increasing and bounded above by 2. This will require two separate proofs by induction - one to show that the sequence is monotone increasing and one to show that it is bounded above by 2.

To show that this sequence is monotone increasing, we must prove that the statement \(P(n) = (a_n \leq a_{n+1})\) is true for all integers \(n \geq 1\).

Since \(a_2 = \sqrt{2} + \sqrt{2}\), we see that \(P(1)\) is the statement

\[\sqrt{2} \leq \sqrt{2} + \sqrt{2}.\]

This statement is true because \(2 < 2 + \sqrt{2}\) which, upon taking the square root of both sides of this inequality, gives us \(\sqrt{2} < \sqrt{2} + \sqrt{2}\).

Next, we show that the truth of \(P(k)\) implies the truth of \(P(k+1)\). Note that \(P(k) = (a_k \leq a_{k+1})\) and \(P(k+1) = (a_{k+1} \leq a_{k+2})\). Assuming that \(P(k)\) is true, we obtain

\[(a_{k+2})^2 = 2 + a_{k+1} \geq 2 + a_k.\]

Taking the square root of both sides of \(2 + a_k \leq (a_{k+2})^2\) gives

\[\sqrt{2} + a_k \leq a_{k+2}\]

which is equivalent to \(a_{k+1} \leq a_{k+2}\). This completes the proof that the sequence \(a_n\) is monotone increasing.

Now, we show that \(a_n\) is bounded above by 2. In other words, we show that the statement \(P(n) = (a_n \leq 2)\) is true for all integers \(n \geq 1\).

The statement \(P(1) = (\sqrt{2} \leq 2)\) is obviously true. Assuming that \(P(k) = (a_k \leq 2)\) is true, we obtain

\[(a_{k+1})^2 = 2 + a_k \leq 2 + 2 = 4\]
which gives us \( a_{k+1} \leq \sqrt{4} = 2 \). This show that the truth of \( P(k) \) implies the truth of \( P(k+1) \).

Since the sequence \( a_n \) is monotone increasing and bounded above, Theorem 4 tells us that \( a_n \) converges to a limit \( L \). Sometimes, even when we know the limit of a sequence exists, it is hard to actually find the limit, but in this example, we can actually do it. Since \( a_{n+1} = \sqrt{2 + a_n} \) for all \( n \geq 1 \), we have

\[
\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{2 + a_n}
\]

which gives us

\[
L = \sqrt{2 + L}.
\]

Since the only solution of this equation is \( L = 2 \), we conclude that \( \lim_{n \to \infty} a_n = 2 \).

**Exercise 14** In the analysis of the sequence in Example 13, we frequently made use of the fact that if \( x \) and \( y \) are real numbers with \( 0 < x < y \), then \( \sqrt{x} < \sqrt{y} \). Prove this fact. (Hint: \( \sqrt{x} \) is, by definition, the unique positive real number, \( z \), such that \( z^2 = x \) and \( \sqrt{y} \) is the unique positive real number, \( w \), such that \( w^2 = y \). Assume, for the sake of obtaining a contradiction, that \( \sqrt{x} > \sqrt{y} \).)

**Exercise 15** In the textbook, Section 1.3, do problems 1 (a through f), 4, 7, 9, 14, and 18.